alt. num. problems of exams 1911-1912

Edward H. Bright.
245 Davis Hall
1915/1916

SOLID GEOMETRY

BY

FLETCHER DURELL, PH.D.

HEAD OF THE MATHEMATICAL DEPARTMENT, THE LAWRENCEVILLE SCHOOL

NEW YORK
CHARLES E. MERRILL CO.

I. M.
Edward Bright

DURELL'S MATHEMATICAL SERIES

ARITHMETIC

TWO BOOK SERIES
- Elementary Arithmetic, 40 cents
  - Teachers' Edition, 50 cents
- Advanced Arithmetic, 64 cents

THREE BOOK SERIES
- Book One, 50 cents
- Book Two, 56 cents
- Book Three, 60 cents

ALGEBRA

TWO BOOK COURSE
- Book One, $1.00
- Book Two

INTRODUCTORY ALGEBRA, 60 cents
SCHOOL ALGEBRA, $1.10

GEOMETRY

PLANE GEOMETRY, 75 cents
SOLID GEOMETRY, 75 cents
PLANE AND SOLID GEOMETRY, $1.25

TRIGONOMETRY

PLANE TRIGONOMETRY AND TABLES, $1.25
PLANE AND SPHERICAL TRIGONOMETRY AND TABLES, $1.40
PLANE AND SPHERICAL TRIGONOMETRY WITH SURVEYING AND TABLES, $1.50
LOGARITHMIC AND TRIGONOMETRIC TABLES, 60 cents

PREFACE

One of the main purposes in writing this book has been to try to present the subject of Geometry so that the pupil shall understand it not merely as a series of correct deductions, but shall realize the value and meaning of its principles as well. This aspect of the subject has been directly presented in some places, and it is hoped that it pervades and shapes the presentation in all places.

Again, teachers of Geometry generally agree that the most difficult part of their work lies in developing in pupils the power to work original exercises. The second main purpose of the book is to aid in the solution of this difficulty by arranging original exercises in groups, each of the earlier groups to be worked by a distinct method. The pupil is to be kept working at each of these groups till he masters the method involved in it. Later, groups of mixed exercises to be worked by various methods are given.

In the current exercises at the bottom of the page, only such exercises are used as can readily be solved in connection with the daily work. All difficult originals are included in the groups of exercises as indicated above.

Similarly, in the writer's opinion, many of the numeri-

cal applications of geometry call for special methods of solution, and the thorough treatment of such exercises should be taken up separately and systematically. [See pp. 304–318, etc.] In the daily extempore work only such numerical problems are included as are needed to make clear and definite the meaning and value of the geometric principles considered.

Every attempt has been made to create and cultivate the heuristic attitude on the part of the pupil. This has been done by the method of initiating the pupil into original work described above, by queries in the course of proofs, and also at the bottom of different pages, and also by occasional queries in the course of the text where definitions and discussions are presented. In the writer's opinion, the time has not yet come for the purely heuristic study of Geometry in most schools, but it is all-important to use every means to arouse in the pupil the attitude and energy of original investigation in the study of the subject.

In other respects, the aim has been to depart as little as possible from the methods most generally used at present in teaching geometry.

The Practical Applications (Groups 88-91) have been drawn from many sources, but the author wishes to express his especial indebtedness to the Committee which has collected the Real Applied Problems published from time to time in *School Science and Mathematics*, and of which Professor J. F. Millis of the Francis W. Parker School of Chicago is the chairman. Page 360 is due almost entirely to Professor William Betz of the East High School of Rochester, N. Y.

FLETCHER DURELL.

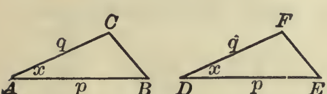
LAWRENCEVILLE, N. J., Sept. 1, 1904.

TO THE TEACHER

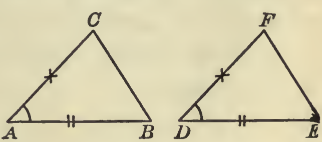
1. In working original exercises, one of the chief difficulties of pupils lies in their inability to construct the figure required and to make the particular enunciation from it. Many pupils, who are quite unable to do this preliminary work, after it is done can readily discover a proof or a solution. In many exercises in this book the figure is drawn and the particular enunciation made. It is left to the discretion of the teacher to determine for what other exercises it is best to do this for pupils.

2. It is frequently important to give partial aid to the pupil by eliciting the outline of a proof by questions such as the following: "On this figure (or, in these two triangles) what angles are equal, and why?" "What lines are equal, and why?" etc.

3. In many cases it is also helpful to mark in colored crayon pairs of equal lines, or of equal angles. Thus, in the figure on p. 37 lines AB and DE may be drawn with red crayon, AC and DF with blue, and the angles A and D marked by small arcs drawn with green crayon. If colored crayons are not at hand, the homologous equal parts may be denoted by like symbols placed on them, thus:

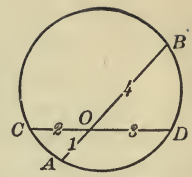


or thus:



In solving theorems concerning proportional lines, it is occasionally helpful to denote the lines in a proportion

(either given or to be proved) by figures denoting the order in which the lines are to be taken. Thus, if $OA : OC = OD : OB$, the relation may be indicated as on the figure.



4. It is sometimes helpful to vary the symbolism of the book. Thus, in dealing with inequalities a convenient symbol for "angle" is $\measuredangle$,

$$\text{as } \measuredangle A > \measuredangle B.$$

5. Each pupil need be required to work only so many originals in each group as will give him a mastery of the particular method involved. A large number of exercises is given in order that the teacher may have many to select from and may vary the work with successive classes.

6. It is important to insist that the solutions of exercises for the first few weeks be carefully written out; later, for many pupils, oral demonstration will be sufficient and ground can be covered more rapidly by its use.

7. In leading pupils to appreciate the meaning of theorems, it is helpful at times to point out that not every theorem has for its object the demonstration of a new and unexpected truth (i. e., not all are "synthetic"), but that some theorems are analytic, it being their purpose to reduce an obvious truth to the certain few principles with which we start in Geometry. Their function is, therefore, to simplify and clarify the subject rather than to extend its content.

REFERENCES TO PLANE GEOMETRY.

DEFINITIONS AND FIRST PRINCIPLES.

41. Parallel lines are lines in the same plane which do not meet, however far they be produced.

GENERAL AXIOMS.

1. *Things which are equal to the same thing, or to equal things, are equal to each other.*
2. *If equals be added to equals, the sums are equal.*
3. *If equals be subtracted from equals, the remainders are equal.*
4. *Doubles of equals are equal; or, in general, if equals be multiplied by equals the products are equal.*
5. *Halves of equals are equal; or, in general, if equals be divided by equals the quotients are equal.*
6. *The whole is equal to the sum of its parts.*
7. *The whole is greater than any of its parts.*
8. *A quantity may be substituted for its equal in any process.*
9. *If equals be added to, or subtracted from, unequals, the results are unequal in the same order; if unequals be added to unequals in the same order, the results are unequal in that order.*
10. *Doubles, or halves, of unequals are unequal in the same order.*
11. *If unequals be subtracted from equals, the remainders are unequal in the reverse order.*

12. *If, of three quantities, the first is greater than the second, and the second is greater than the third, then the first is greater than the third.*

GEOMETRIC AXIOMS.

1. *Through two given points only one straight line can be passed.*

2. *A geometric figure may be freely moved in space without any change in form or size.*

3. *Through a given point one straight line and only one can be drawn parallel to another given straight line.*

Geometric figures which coincide are equal.

71. *At a given point in a straight line but one perpendicular can be erected to the line.*

75. *The complements of two equal angles are equal; the supplements of two equal angles are equal.*

76. *The sum of all the angles about a point equals four right angles.*

BOOK I.

78. *If one straight line intersects another straight line, the opposite or vertical angles are equal.*

79. *If, from a point in a perpendicular to a given line, two oblique lines be drawn cutting off on the given line equal segments from the foot of the perpendicular, the oblique lines are equal and make equal angles with the perpendicular.*

81. A **triangle** is a portion of a plane bounded by three straight lines.

92. *The sum of any two sides of a triangle is greater than the third side.*

94. *The perpendicular is the shortest line that can be drawn from a given point to a given line.*

95. *If, from a point within a triangle, two lines be drawn to the extremities of one side of the triangle, the sum of the*

other two sides of the triangle is greater than the sum of the two lines so drawn.

96. *Two triangles are equal if two sides and the included angle of one are equal, respectively, to two sides and the included angle of the other.*

98. *Two right triangles are equal if the hypotenuse and an acute angle of one are equal to the hypotenuse and an acute angle of the other.*

99. *In an isosceles triangle the angles opposite the equal sides are equal.*

100. *If two angles of a triangle are equal, the sides opposite are equal, and the triangle is isosceles.*

101. *Two triangles are equal if three sides of one are equal to three sides of the other, respectively.*

102. *Two right triangles are equal if the hypotenuse and a leg of one are equal to the hypotenuse and a leg of the other.*

108. *If two sides of a triangle are equal, respectively, to two sides of another triangle, but the third side of the first is greater than the third side of the second, then the angle opposite the third side of the first triangle is greater than the angle opposite the third side of the second.*

109. *Of lines drawn from the same point in a perpendicular, and cutting off unequal segments from the foot of the perpendicular, the more remote is the greater.*

112. I. *Every point in the perpendicular bisector of a line is equally distant from the extremities of the line; and*

II. *Every point not in the perpendicular bisector is unequally distant from the extremities of the line.*

113. *Two points each equidistant from the extremities of a line determine the perpendicular bisector of the line.*

115. *The perpendicular bisector of a line is the locus of all points equidistant from the extremities of the line.*

120. **Parallel lines** are lines in the same plane which do not meet, however far they be produced.

121. *Two straight lines in the same plane, perpendicular to the same straight line, are parallel.*

122. *Two straight lines parallel to a third straight line are parallel to each other;*

Lines parallel to parallel lines are parallel;

Lines perpendicular to parallel lines are parallel;

Lines perpendicular to non-parallel lines are not parallel.

123. *If a straight line is perpendicular to one of two given parallel lines it is perpendicular to the other also.*

125. *If two straight lines are cut by a transversal, making the alternate interior angles equal, the two straight lines are parallel.*

127. *If two straight lines are cut by a transversal, making the exterior interior angles equal, the two straight lines are parallel.*

130. *Two angles whose sides are parallel, each to each, are either equal or supplementary.*

134. *The sum of the angles of a triangle is equal to two right angles.*

142. *Two right triangles are equal if a leg and an acute angle of one are equal to a leg and the homologous acute angle of the other.*

147. A **parallelogram** is a quadrilateral whose opposite sides are parallel.

149. A **rhombus** is a rhomboid whose sides are equal.

150. A **rectangle** is a parallelogram whose angles are right angles.

151. A **square** is a rectangle whose sides are equal.

155. *The opposite sides of a parallelogram are equal, and its opposite angles are also equal.*

156. *A diagonal divides a parallelogram into two equal triangles.*

157. *Parallel lines comprehended between parallel lines are equal.*

160. *If two sides of a quadrilateral are equal and parallel, the other two sides are equal and parallel and the figure is a parallelogram.*

162. *Two parallelograms are equal if two adjacent sides and the included angle of one are equal, respectively, to two adjacent sides and the included angle of the other.*

163. *Two rectangles which have equal bases and equal altitudes are equal.*

174. *In an equiangular polygon of n sides each angle equals* $\frac{(n-2)2\ rt. \angle s}{n}$, or $\frac{2n-4}{n} rt. \angle s.$

175. *The sum of the exterior angles of a polygon formed by producing its sides in succession equals four right angles.*

176. *If three or more parallels intercept equal parts on one transversal, they intercept equal parts on every transversal.*

177. *The line which joins the midpoints of two sides of a triangle is parallel to the third side, and is equal to one-half the third side.*

178. *The line which bisects one side of a triangle and is parallel to another side bisects the third side.*

179. *The line which joins the midpoints of the legs of a trapezoid is parallel to the bases and equal to one-half their sum.*

BOOK II.

197. A **circle** is a portion of a plane bounded by a curved line, all points of which are equally distant from a point within called the **center.**

198. A **radius** of a circle is a straight line drawn from the center to any point on the circumference.

208. *Radii of the same circle, or of equal circles, are equal.*

216. *In the same circle, or in equal circles, equal arcs subtend equal angles at the center.*

217. *In the same circle, or in equal circles, of two unequal central angles the greater angle intercepts the greater arc, and, conversely, of two unequal arcs, the greater arc subtends the greater angle at the center.*

218. *In the same circle, or in equal circles, equal chords subtend equal arcs.*

219. *In the same circle, or in equal circles, equal arcs are subtended by equal chords.*

220. *In the same circle, or in equal circles, the greater of two (minor) arcs is subtended by the greater chord; and,* CONVERSELY, *the greater of two chords subtends the greater (minor) arc.*

226. *In the same circle, or in equal circles, equal chords are equidistant from the center; and,* CONVERSELY, *chords which are equidistant from the center are equal.*

229. *A straight line perpendicular to a radius at its extremity is tangent to the circle.*

230. *The radius drawn to the point of contact is perpendicular to a tangent to a circle.*

237. *The two tangents drawn to a circle from a point outside the circle are equal, and make equal angles with a line drawn from the point to the center.*

241. *If two circles intersect, their line of centers is perpendicular to their common chord at its middle point.*

253. Properties of variables and limits.

1. *The limit of the sum of a number of variables equals the sum of the limits of these variables.*

2. *The limit of* a *times a variable equals* a *times the limit of the variable,* a *being a constant.*

3. *The limit of* $\frac{1}{a}$*th part of variable is* $\frac{1}{a}$*th part of the limit of the variable,* a *being a constant.*

254. *If two variables are always equal, and each approaches a limit, their limits are equal.*

257. *The number of degrees in a central angle equals the number of degrees in the intercepted arc;* that is, *a central angle is measured by its intercepted arc.*

273. *From a given point without a given line to draw a perpendicular to the line.*

274. *At a given point in a given line erect a perpendicular to that line.*

279. *Through a given point without a given straight line to draw a line parallel to a given line.*

286. *To circumscribe a circle about a given triangle.*

BOOK III.

303. *The mean proportional between two quantities is equal to the square root of their product.*

305. *If the antecedents of a proportion are equal, the consequents are equal.*

307. *If four quantities are in proportion, they are in proportion by* **alternation**; *that is, the first term is to the third as the second is to the fourth.*

310. *If four quantities are in proportion, they are in proportion by* **division**; *that is, the difference of the first two is to the second as the difference of the last two is to the last.*

312. *In a series of equal ratios, the sum of all the antecedents is to the sum of all the consequents as any one antecedent is to its consequent.*

314. *Like powers, or like roots, of the terms of a proportion are in proportion.*

317. *A line parallel to one side of a triangle and meeting the other two sides, divides these sides proportionally.*

318. *If a line parallel to the base cut the sides of a triangle, a side is to a segment of that side as the other side is to the corresponding segment of the second side.*

321. **Similar polygons** are polygons having their ho-

mologous angles equal and their homologous sides proportional.

323. *If two triangles are mutually equiangular, they are similar.*

326. *If two triangles have their homologous sides proportional they are similar.*

327. *If two triangles have an angle of one equal to an angle of the other, and the including sides proportional, the triangles are similar.*

328. *If two triangles have their sides parallel, or perpendicular, each to each, the triangles are similar.*

329. *If two polygons are similar, they may be separated into the same number of triangles, similar, each to each, and similarly placed.*

342. *In a right triangle,*

I. *The altitude to the hypotenuse is a mean proportional between the segments of the hypotenuse;*

II. *Each leg is a mean proportional between the hypotenuse and the segment of the hypotenuse adjacent to the given leg.*

343. *The perpendicular to the diameter from any point in the circumference of a circle is a mean proportional between the segments of the diameter; and the chord joining the point to an extremity of the diameter is a mean proportional between the diameter and the segment of the diameter adjacent to the chord.*

346. *In a right triangle, the square of the hypotenuse is equal to the sum of the squares of the legs.*

347. *In a right triangle, the square of either leg is equal to the square of the hypotenuse minus the square of the other leg.*

351. *If the square on the side of a triangle equals the sum of the squares on the other two sides, the angle opposite the first side is a right angle.*

352. *If, in any triangle, a median be drawn to one side,*

I. *The sum of the squares of the other two sides is equal to twice the square of half the given side, increased by twice the square of the median upon that side;* and

II. *The difference of the squares of the other two sides is equal to twice the product of the given side by the projection of the median upon that side.*

BOOK IV.

383. *The area of a rectangle is equal to the product of its base by its altitude.*

385. *The area of a parallelogram is equal to the product of its base by its altitude.*

389. *The area of a triangle is equal to one-half the product of its base by its altitude.*

390. *Triangles which have equal bases and equal altitudes (or which have equal bases and their vertices in a line parallel to the base) are equivalent.*

391. *Triangles which have equal bases are to each other as their altitudes;*

Triangles which have equal altitudes are to each other as their bases.

392. *Any two triangles are to each other as the products of their bases and altitudes.*

397. *If two triangles have an angle of one equal to an angle of the other, their areas are to each other as the products of the sides including the equal angles.*

398. *The areas of any two similar triangles are to each other as the squares of any two homologous sides.*

399. *The areas of two similar polygons are to each other as the squares of any two homologous sides.*

BOOK V.

444. Formula for the circumference in terms of the radius.

$$C = 2\pi R.$$

449. *The area of a circle is equal to one-half the product of its circumference by its radius.*

486. Two points symmetrical with respect to a line or axis are points such that the straight line joining them is bisected by the given line at right angles.

487. A figure symmetrical with respect to an axis is a figure such that each point in the one part of the figure has a point in the other part symmetrical to the given point, with respect to an axis.

489. Two points symmetrical with respect to a point or center are points such that the straight line joining them is bisected by the point or center.

490. A figure symmetrical with respect to a point or center is a figure such that each point in the figure has another point in the figure symmetrical to the given point with respect to the center.

TABLE OF CONTENTS

SYMBOLS AND ABBREVIATIONS

$+$ *plus*, or *increased by*.
$-$ *minus*, or *diminished by*.
$\times$ *multiplied by*.
$\div$ *divided by*.
$=$ *equals; is* (or *are*) *equal to*.
$\doteq$ *approaches* (*as a limit*).
$\backsimeq$ *is* (or *are*) *equivalent to*.
$>$ *is* (or *are*) *greater than*.
$<$ *is* (or *are*) *less than*.
$\therefore$ *therefore*.
$\perp$ *perpendicular*, *perpendicular to*, or, *is perpendicular to*.
$\perp$s *perpendiculars*.
$\parallel$ *parallel*, or, *is parallel to*.
$\parallel$s *parallels*.
$\angle$, $\angle$s *angle, angles*
$\triangle$, $\triangle$s *triangle, triangles*.
▱, ▱s *parallelogram, parallelograms*.
$\odot$, $\odot$s *circle, circles*.

Adj. . . *adjacent*.
Alt. . . *alternate*.
Art. . . *article*.
Ax. . . . *axiom*.
Constr. . *construction*.
Cor. . . *corollary*.
Def. . . *definition*.
Ex. . . . *exercise*.
Ext. . . *exterior*.
Fig. . . *figure*.
Hyp. . . *hypothesis*.
Ident. . *identity*.
Int. . . *interior*.
Post. . . *postulate*.
Prop. . . *proposition*.
Rt. . . . *right*.
Sug. . . *suggestion*.
Sup. . . *supplementary*.
St. . . . *straight*.

Q. E. D. quod erat demonstrandum; that is, which was to be proved.
Q. E. F. quod erat faciendum; that is, which was to be made.

A few other abbreviations and symbols will be introduced and their meaning indicated later on.

SOLID GEOMETRY

Book VI

LINES, PLANES AND ANGLES IN SPACE

DEFINITIONS AND FIRST PRINCIPLES

497. Solid Geometry treats of the properties of space of three dimensions.

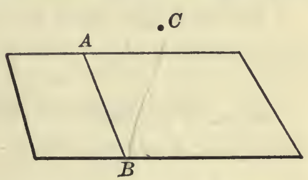
Many of the properties of space of three dimensions are determined by use of the plane and of the properties of plane figures already obtained in Plane Geometry.

498. A **plane** is a surface such that, if any two points in it be joined by a straight line, the line lies wholly in the surface.

499. A **plane is determined** by given points or lines, if no other plane can pass through the given points or lines without coinciding with the given plane.

500. Fundamental property of a plane in space. *A plane is determined by any three points not in a straight line.*

For, if through a line connecting two given points, A and B, a plane be passed, the plane, if rotated, can pass through a third given point, C, in but one position.

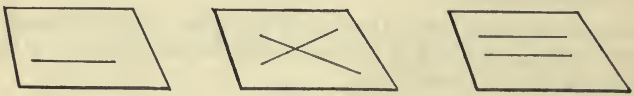


The importance of the above principle is seen from the fact that it reduces an unlimited surface to three points, thus making a vast economy to the attention. It also enables us to connect different planes, and treat of their properties systematically.

501. Other modes of determining a plane. *A plane may also be determined* by any equivalent of three points not in a straight line, as by

a straight line and a point outside the line; or by
two intersecting straight lines; or by
two parallel straight lines.

It is often more convenient to use one of these latter methods of determining a plane than to reduce the data to three points and use Art. 500.



502. Representation of a plane in geometric figures. In reasoning concerning the plane, it is often an advantage to have the plane represented in all directions. Hence, in drawing a geometric figure, a plane is usually represented to the eye by a small parallelogram.

This is virtually a double use of two intersecting lines, or of two parallel lines, to determine a plane (Art. 501).

503. Postulate of Solid Geometry. The principle of Art. 499 may also be stated as a postulate, thus:

Through any three points not in a straight line (*or their equivalent*) *a plane may be passed.*

504. The **foot** of a line is the point in which the line intersects a given plane.

505. A straight line perpendicular to a plane is a line perpendicular to every line in the plane drawn through its foot.

A straight line perpendicular to a plane is sometimes called a **normal** to the plane.

506. **A parallel straight line and plane** are a line and plane which cannot meet, however far they be produced.

507. **Parallel planes** are planes which cannot meet, however far they be produced.

508. Properties of planes inferred immediately.

1. *A straight line, not in a given plane, can intersect the given plane in but one point.*

For, if the line intersect the given plane in two or more points, by definition of a plane, the line must lie in the plane. Art. 498.

2. *The intersection of two planes is a straight line.*

For, if two points common to the two planes be joined by a straight line, this line lies in each plane (Art. 498); and no other point can be common to the two planes, for, through a straight line and a point outside of it only one plane can be passed. Art. 501.

Ex. 1. Give an example of a plane surface; of a curved surface; of a surface, part plane and part curved; of a surface composed of different plane surfaces.

Ex. 2. Four points, not all in the same plane determine how many different planes? how many different straight lines?

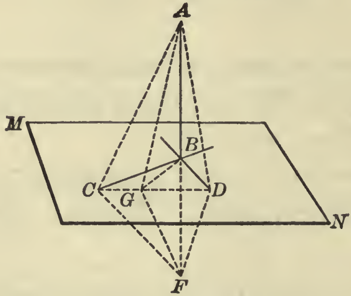
Ex. 3. Three parallel straight lines, not in the same plane, determine how many different planes?

Ex. 4. Four parallel straight lines can determine how many different planes?

Ex. 5. Two intersecting straight lines and a point, not in their plane, determine how many different planes?

Proposition I. Theorem

509. *If a straight line is perpendicular to each of two other straight lines at their point of intersection, it is perpendicular to the plane of those lines.*



Given $AB \perp$ lines BC and BD, and the plane MN passing through BC and BD.

To prove $AB \perp$ plane MN.

Proof. Through B draw BG, any other line in the plane MN.

Draw any convenient line CD intersecting BC, BG and BD in the points C, G and D, respectively.

Produce the line AB to F, making $BF = AB$.

Connect the points C, G, D with A, and also with F.

Then, in the $\triangle$s ACD and FCD, $CD = CD$. Ident.

$AC = CF$, and $AD = DF$. Art. 112.

$\therefore \triangle ACD = \triangle FCD$. (Why?)

$\therefore \angle ACD = \angle FCD$. (Why?)

Then, in the $\triangle$s ACG and FCG, $CG = CG$, (Why?)

$AC = CF$, and $\angle ACG = \angle FCG$. (Why?)

$\therefore \triangle ACG = \triangle FCG$. (Why?)

$\therefore AG = GF.$ (Why?)

$\therefore B$ and G are each equidistant from the points A and F.

$\therefore BG$ is $\perp AF$; that is, $AB \perp BG$. Art. 113.

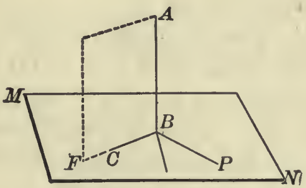
$\therefore AB \perp$ plane MN, Art. 505.

(*for it is $\perp$ any line, BG, in the plane MN, through its foot*).

Q. E. D.

Proposition II. Theorem

510. *All the perpendiculars that can be drawn to a given line at a given point in the line lie in a plane perpendicular to the line at the given point.*



Given the plane MN and the line BC both $\perp$ line AB at the point B.

To prove that BC lies in the plane MN.

Proof. Pass a plane AF through the intersecting lines AB and BC. Art. 503.

This plane will intersect the plane MN in a straight line BF. Art. 508, 2.

But $AB \perp$ plane MN (Hyp.) $\therefore AB \perp BF$. Art. 505.

Also $AB \perp BC$. Hyp.

$\therefore$ in the plane AF, BC and $BF \perp AB$ at B.

$\therefore BC$ and BF coincide. Art. 71.

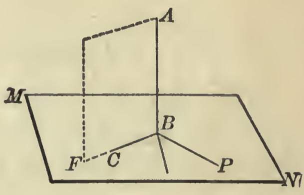
But BF is in the plane MN.

$\therefore BC$ must be in the plane MN,

(*for BC coincides with BF, which lies in the plane MN*).

Q. E. D.

511. Cor. 1. *At a given point B in the straight line AB, to construct a plane perpendicular to the line AB.* Pass a plane AF through AB in any convenient direction, and in the plane AF at the point B construct $BF \perp AB$ (Art. 274). Pass another plane through AB, and in it construct $BP \perp AB$. Through the lines BF and BP pass the plane MN (Art. 503). MN is the required plane (Art. 509).



512. Cor. 2. *Through a given external point, P, to pass a plane perpendicular to a given line, AB.* Pass a plane through AB and P (Art. 503), and in this plane draw $PB \perp AB$ (Art. 273). Pass another plane through AB, as AF, and in AF draw $BF \perp AB$ at B (Art. 274). Pass a plane through BP and BF (Art. 503). This will be the plane required (Art. 509).

513. Cor. 3. *Through a given point but one plane can be passed perpendicular to a given line.*

Ex. 1. Five points, no four of which are in the same plane, determine how many different planes? how many different straight lines?

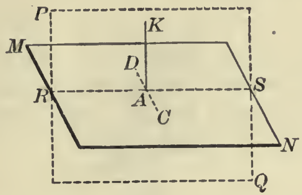
Ex. 2. A straight line and two points, not all of which are in the same plane, determine how many different planes?

Ex. 3. In the figure, p. 322, prove that the triangles GAD and GDF are equal.

Ex. 4. In the same figure, if $AB=8$ and $BC=6$, find FC.

Proposition III. Problem

514. *At a given point in a plane, to erect a perpendicular to the plane.*



Given the point A in plane MN.

To construct a line perpendicular to MN at the point A.

Construction. Through the point A draw any line CD in the plane MN.

Also through the point A pass the plane $PQ \perp CD$ (Art. 511), intersecting the plane MN in the line RS. Art. 508, 2.

In the plane PQ draw $AK \perp$ line RS at A. Art. 274.

Then AK is the $\perp$ required.

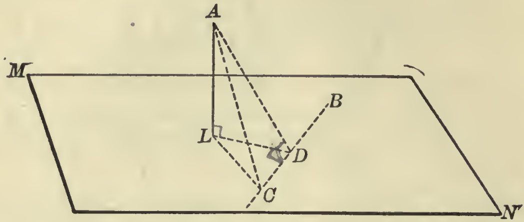
Proof.	$CD \perp$ plane PQ.	Constr.
	$\therefore CD \perp AK$.	Art. 505.
Hence	$AK \perp CD$.	
But	$AK \perp RS$.	Constr.
	$\therefore AK \perp$ plane MN.	Art. 509.

Q. E. F.

515. Cor. *At a given point in a plane but one perpendicular to the plane can be drawn.* For, if two $\perp$s could be drawn at the given point, a plane could be passed through them intersecting the given plane. Then the two $\perp$s would be in the new plane and $\perp$ to the same line (the line of intersection of the two planes, Art. 505), which is impossible (Art. 71).

Proposition IV. Problem

516. *From a given point without a plane, to draw a line perpendicular to the plane.*



Given the plane MN and the point A external to it.

To construct from A a line $\perp$ plane MN.

Construction. In the plane MN draw any convenient line BC. Pass a plane through BC and A (Art. 503), and in this plane draw $AD \perp BC$. Art. 273.

In the plane MN draw $LD \perp BC$. Art. 274.

Pass a plane through AD and LD (Art. 503), and in that plane draw $AL \perp LD$. Art. 273.

Then AL is the $\perp$ required.

Proof. Take any point C in BC except D, and draw LC and AC.

Then $\triangle$ ADC, ADL and LDC are right $\triangle$. Constr.

$\therefore \overline{AC}^2 = \overline{AD}^2 + \overline{DC}^2$. Art. 400.

$\therefore \overline{AC}^2 = \overline{AL}^2 + \overline{LD}^2 + \overline{DC}^2$. Art. 400, Ax. 8.

$\therefore \overline{AC}^2 = \overline{AL}^2 + \overline{LC}^2$. Art. 400, Ax. 8.

$\therefore \angle ALC$ is a right $\angle$. Art. 351.

But $AL \perp LD$. Constr.

$\therefore AL \perp MN$. Art. 509.

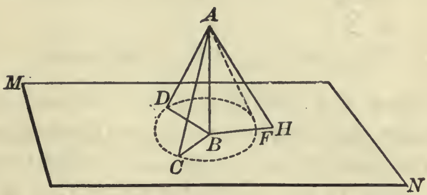
Q. E. F.

517. Cor. *But one perpendicular can be drawn from a given external point to a given plane.*

PROPOSITION V. THEOREM

518. I. *Oblique lines drawn from a point to a plane, meeting the plane at equal distances from the foot of the perpendicular, are equal;*

II. *Of two oblique lines drawn from a point to a plane, but meeting the plane at unequal distances from the foot of the perpendicular, the more remote is the greater.*



Given $AB \perp$ plane MN, $BD = BC$, and $BH > BC$.

To prove $AD = AC$, and $AH > AC$.

Proof. I. In the right $\triangle$ ABD and ABC,

$AB = AB$, and $BD = BC$. (Why?)

$\therefore \triangle ABD = \triangle ABC$. (Why?)

$\therefore AD = AC$. (Why?)

II. On BH take $BF = BC$, and draw AF.

Then $AF = AC$ (*by part of theorem just proved*).

But $AH > AF$. (Why?)

$\therefore AH > AC$. Ax. 8.

Q. E. D.

519. COR. 1. CONVERSELY: *Equal oblique lines drawn from a point to a plane meet the plane at equal distances from the foot of the perpendicular drawn from the same point to the plane; and, of two unequal lines so drawn, the greater line meets the plane at the greater distance from the foot of the perpendicular.*

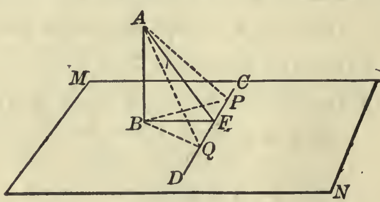
520. Cor. 2. *The locus of a point in space equidistant from all the points in the circumference of a circle is a straight line passing through the center of the circle and perpendicular to its plane.*

521. Cor. 3. *The perpendicular is the shortest line that can be drawn from a given point to a given plane.*

522. Def. The **distance from a point to a plane** is the perpendicular drawn from the point to the plane.

Proposition VI. Theorem

523. *If from the foot of a perpendicular to a plane a line be drawn at right angles to any line in the plane, the line drawn from the point of intersection so formed to any point in the perpendicular, is perpendicular to the line of the plane.*



Given $AB \perp$ plane MN, and $BF \perp CD$, any line in MN.

To prove $AF \perp CD$.

Proof. On CD take FP and FQ equal segments.

Draw AP, BP, AQ, BQ.

Then $BP = BQ$. Art. 112.

Hence $AP = AQ$. Art. 518.

$\therefore$ in the line AF, the point A is equidistant from P and Q, and F is equidistant from P and Q.

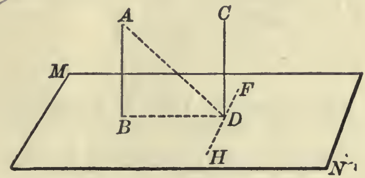
$\therefore AF \perp CD$. (Why?)

Q. E. D.

Ex. In the above figure, if $AB=6$, $AF=8$, and $AQ=10$, find QF, BF and BQ.

Proposition VII. Theorem

524. *Two straight lines perpendicular to the same plane are parallel.*



Given the lines AB and $CD \perp$ plane MN.

To prove $AB \parallel CD$.

Proof. Draw BD, and through D, in the plane MN, draw $FH \perp BD$.

Draw AD.

Then $BD \perp FH$. Constr.

$AD \perp FH$. Art. 523.

$CD \perp FH$. Art. 505.

$\therefore BD$, AD and CD are all $\perp FH$ at the point D.

$\therefore BD$, AD and CD all lie in the same plane. Art. 510.

$\therefore AB$ and CD are in the same plane. (Why?)

But AB and CD are $\perp BD$. Art. 505.

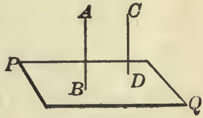
$\therefore AB$ and CD are $\parallel$. Art. 121.

Q. E. D.

525. Cor. 1. *If one of two parallel lines is perpendicular to a plane, the other is perpendicular to the plane also.*

For, if AB and CD be $\parallel$, and $AB \perp$ plane PQ, a line drawn from $C \perp PQ$ must be $\parallel AB$. Art. 524.

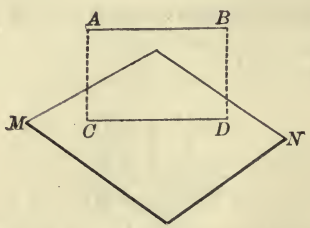
But CD must coincide with the line so drawn (Art. 47, 3); $\therefore CD \perp PQ$.



526. Cor. 2. *If two straight lines are each parallel to a third straight line, they are parallel to each other.* For, if a plane be drawn ⊥ to the third line, each of the two other lines must be ⊥ to it (Art. 525), and therefore be ∥ to each other (Art. 524).

Proposition VIII. Theorem

527. *If a straight line external to a given plane is parallel to a line in the plane, then the first line is parallel to the given plane.*



Given the straight line AB ∥ line CD in the plane MN.

To prove AB ∥ plane MN.

Proof. Pass a plane through the ∥ lines AB and CD.

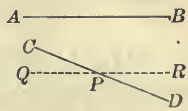
If AB meets MN it must meet it in the line CD.

But AB and CD cannot meet, for they are ∥. Art. 120.

∴ AB and MN cannot meet and are parallel. Art. 506.

Q. E. D.

528. Cor. 1. *If a straight line is parallel to a plane, the intersection of the plane with any plane passing through the given line is parallel to the given line.*



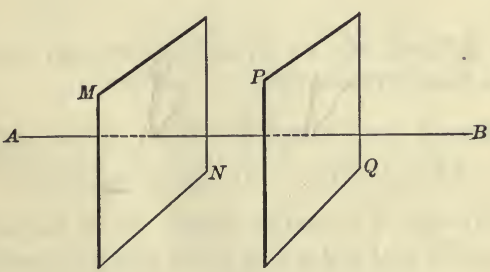
529. Cor. 2. *Through a given line* (CD) *to pass a plane parallel to another given line* (AB).

Through P, any point in CD, draw $QR \parallel AB$ (Art. 279). Through CD and QR pass a plane (Art. 503). This will be the plane required (Art. 527).

If AB and CD are not parallel, but one plane can be drawn through $CD \parallel AB$.

Proposition IX. Theorem

530. *Two planes perpendicular to the same straight line are parallel.*



Given the planes MN and $PQ \perp$ line AB.

To prove $MN \parallel PQ$.

Proof. If MN and PQ are not parallel, on being produced they will meet.

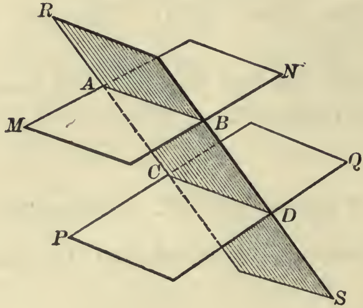
We shall then have two planes drawn from a point perpendicular to a given line, which is impossible. Art. 513.

$\therefore$ MN and PQ are parallel. Art. 507.

Q. E. D.

Proposition X. Theorem

531. *If two parallel planes are cut by a third plane, the intersections are parallel lines.*



Given MN and PQ two $\parallel$ planes intersected by the plane RS in the lines AB and CD.

To prove $AB \parallel CD$.

Proof. AB and CD lie in the same plane RS.

Also AB and CD cannot meet; for if they did meet the planes MN and PQ would meet, which is impossible. Art. 507.

$\therefore$ AB and CD are parallel. Art. 41.

Q. E. D.

532. Cor. 1. *Parallel lines included between parallel planes are equal.* For, if AC and BD are two parallel lines, a plane may be passed through them (Art. 503), intersecting MN and PQ in the $\parallel$ lines AB and CD. Art. 531.

$\therefore$ $ABDC$ is a parallelogram. Art. 147.

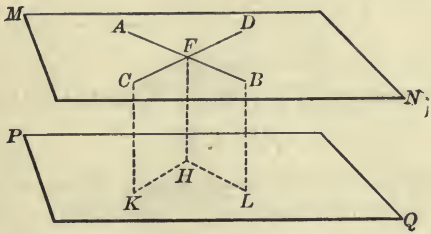
$\therefore$ $AC = BD$. Art. 155.

533. Cor. 2. *Two parallel planes are everywhere equidistant.*

For lines $\perp$ to one of them are $\parallel$ (Art. 524). Hence the segments of these lines included between the $\parallel$ planes are equal (Art. 532).

Proposition XI. Theorem

534. *If two intersecting lines are each parallel to a given plane, the plane of these lines is parallel to the given plane.*



Given the lines AB and CD, each $\parallel$ plane PQ, and intersecting in the point F; and MN a plane through AB and CD.

To prove $MN \parallel PQ$.

Proof. From the point F draw $FH \perp PQ$.

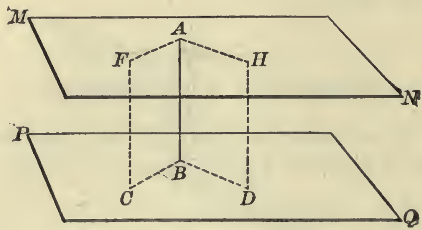
Pass a plane through FC and FH, intersecting PQ in HK; also pass a plane through FB and FH, intersecting PQ in HL.

Then	$HK \parallel FC$, and $HL \parallel FB$.	Art. 528.
But	$FH \perp HK$ and HL.	Art. 505.
	$\therefore FH \perp FC$ and FB.	Art. 123.
	$\therefore FH \perp MN$.	Art. 509.
	$\therefore MN \parallel PQ$.	Art. 530.

Q. E. D.

Proposition XII. Theorem

535. *A straight line perpendicular to one of two parallel planes is perpendicular to the other also.*



Given the plane $MN \parallel$ plane PQ, and $AB \perp PQ$.

To prove $AB \perp MN$.

Proof. Through AB pass a plane intersecting PQ and MN in the lines BC and AF, respectively; also through AB pass another plane intersecting PQ and MN in BD and AH, respectively.

Then $BC \parallel AF$, and $BD \parallel AH$. Art. 531.

But $AB \perp BC$ and BD. Art. 505.

$\therefore AB \perp AF$ and AH. Art. 123.

$\therefore AB \perp$ plane MN. Art. 509.

Q. E. D.

536. Cor. 1. *Through a given point to pass a plane parallel to a given plane.*

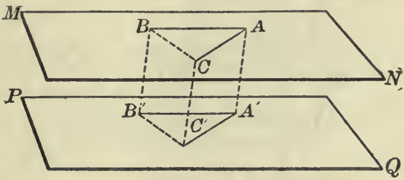
Let the pupil supply the construction.

537. Cor. 2. *Through a given point but one plane can be passed parallel to a given plane.*

Write out.

PROPOSITION XIII. THEOREM

538. *If two angles not in the same plane have their corresponding sides parallel and extending in the same direction, the angles are equal and their planes are parallel.*



Given the $\angle BAC$ in the plane MN, and the $\angle B'A'C'$ in the plane PQ; AB and $A'B'$ $\parallel$ and extending in the same direction; and AC and $A'C'$ $\parallel$ and extending in the same direction.

To prove $\angle BAC = \angle B'A'C'$, and plane $MN \parallel$ plane PQ.

Proof. Take $AB = A'B'$, and $AC = A'C'$.

Draw AA', BB', CC', BC, $B'C'$.

Then $ABB'A'$ is a $\square$, — Art. 160.

(*for AB and A′B′ are = and* $\parallel$).

$\therefore BB'$ and AA' are $=$ and $\parallel$. — Art. 155.

In like manner CC' and AA' are $=$ and $\parallel$.

$\therefore BB'$ and CC' are $=$ and $\parallel$. — (Why?)

$\therefore BCC'B'$ is a $\square$, and $BC = B'C'$. — (Why?)

$\therefore \triangle ABC = \triangle A'B'C'$. — (Why?)

$\therefore \angle A = \angle A'$. — (Why?)

Also $AB \parallel A'B'$, $\therefore AB \parallel$ plane PQ. — Art. 527.

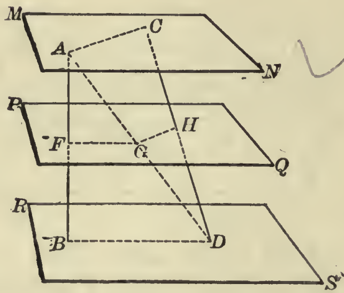
Similarly $AC \parallel$ plane PQ.

$\therefore$ plane $MN \parallel$ plane PQ. — Art. 534.

Q. E. D.

Proposition XIV. Theorem

539. *If two straight lines are intersected by three parallel planes, the corresponding segments of these lines are proportional.*



Given the straight lines AB and CD intersected by the $\parallel$ planes MN, PQ and RS in the points A, F, B, and C H, D, respectively.

To prove $$\frac{AF}{FB}=\frac{CH}{HD}.$$

Proof. Draw the line AD intersecting the plane PQ in G.

Draw FG, BD, GH, AC.

Then $FG \parallel BD$, and $GH \parallel AC$. Art. 531.

$$\therefore \frac{AF}{FB}=\frac{AG}{GD}.$$ Art. 317.

And $$\frac{CH}{HD}=\frac{AG}{GD}.$$ (Why?)

$$\therefore \frac{AF}{FB}=\frac{CH}{HD}.$$ (Why?)

Q. E. D.

Ex. 1. In above figure, if $AF=2$, $FB=5$, and $CH=3$, find CD.

Ex. 2. If $CH=3$, $HD=4$, and $AB=10$, find AF and BF.

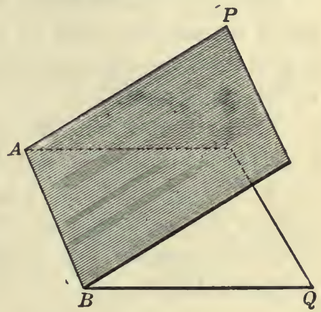
DIHEDRAL ANGLES

540. A **dihedral angle** is the opening between two intersecting planes.

From certain points of view, a dihedral angle may be regarded as a wedge or slice of space cut out by the planes forming the dihedral angle.

541. The **faces** of a dihedral angle are the planes forming the dihedral angle.

The **edge** of a dihedral angle is the straight line in which the faces intersect.



542. Naming dihedral angles. A dihedral angle may be named, or denoted, by naming its edge, as the dihedral angle AB; or by naming four points, two on the edge and one on each face, those on the edge coming between the points on the faces, as $P\text{–}AB\text{–}Q$. The latter method is necessary in naming two or more dihedral angles which have a common edge.

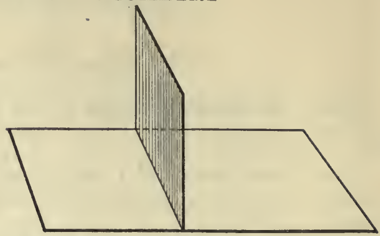
543. Equal dihedral angles are dihedral angles which can be made to coincide.

544. Adjacent dihedral angles are dihedral angles having the same edge and a face between them in common.

545. Vertical dihedral angles are two dihedral angles having the same edge, and the faces of one the prolongations of the faces in the other.

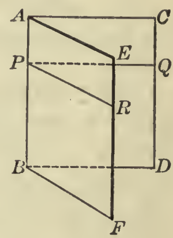
546. A **right dihedral angle** is one of two equal adjacent dihedral angles formed by two planes.

547. A **plane perpendicular** to a given plane is a plane forming a right dihedral angle with the given plane.



Many of the properties of dihedral angles are obtained most conveniently by using a plane angle to represent the dihedral angle.

548. The **plane angle of a dihedral angle** is the angle formed by two lines drawn one in each face, perpendicular to the edge at the same point.



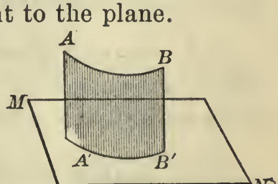
Thus, in the dihedral angle $C\text{–}AB\text{–}F$, if PQ is a line in the face AD perpendicular to the edge AB at P, and PR is a line in face AF perpendicular to the edge AB at P, the angle QPR is the plane angle of the dihedral angle $C\text{–}AB\text{–}F$.

549. Property of plane angles of a dihedral angle.

The magnitude of the plane angle of a dihedral angle is the same at every point of the edge. For let EAC be the plane $\angle$ of the dihedral $\angle$ $E\text{–}AB\text{–}D$ at the point A.
Then $PR \parallel AE$, and $PQ \parallel AC$ (Art. 121.)
$\therefore$ $\angle RPQ = \angle EAC$ (Art. 538).

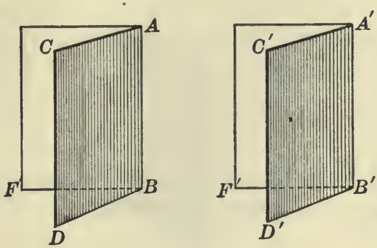
550. The **projection of a point upon a plane** is the foot of a perpendicular drawn from the point to the plane.

551. The **projection of a line upon a plane** is the locus of the projections of all the points of the line on the plane. Thus $A'B'$ is the projection of AB on the plane MN.



X PROPOSITION XV. THEOREM

552. *Two dihedral angles are equal if their plane angles are equal.*



Given $\angle DBF$ the plane $\angle$ of the dihedral $\angle C\text{-}AB\text{-}F$, $\angle D'B'F'$ the plane $\angle$ of the dihedral $\angle C'\text{-}A'B'\text{-}F'$, and $\angle DBF = \angle D'B'F'$.

To prove $\angle C\text{-}AB\text{-}F = \angle C'\text{-}A'B'\text{-}F'$.

Proof. Apply the dihedral $\angle C'\text{-}A'B'\text{-}F'$ to $\angle C\text{-}AB\text{-}F$ so that $\angle D'B'F'$ coincides with its equal, $\angle DBF$. Geom. Ax. 2.

Then line $A'B'$ must coincide with AB, Art. 515.
(*for $A'B'$ and AB are both $\perp$ plane DBF at the point B*).

Hence the plane $A'B'D'$ will coincide with plane ABD, Art. 501.
(*through two intersecting lines only one plane can be passed*).

Also the plane $A'B'F'$ will coincide with the plane ABF,
(*same reason*).

$\therefore$ $\angle C'\text{-}A'B'\text{-}F'$ coincides with $\angle C\text{-}AB\text{-}F$ and is equal to it. Art. 47.

Q. E. D.

553. COR. *The vertical dihedral angles formed by two intersecting planes are equal.*

In like manner, many other properties of plane angles are true of dihedral angles.

Proposition XVI. Theorem

554. *Two dihedral angles have the same ratio as their plane angles.*

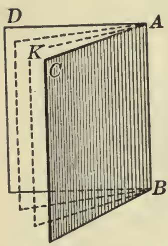


Fig. 1

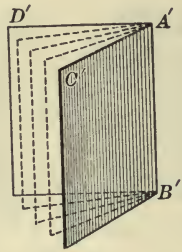


Fig. 2

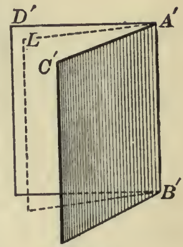


Fig. 3

Given the dihedral $\angle s$ $C\text{–}AB\text{–}D$ and $C'\text{–}A'B'\text{–}D'$ having the plane $\angle s$ CAD and $C'A'D'$, respectively.

To prove $\angle C'\text{–}A'B'\text{–}D' : \angle C\text{–}AB\text{–}D = \angle C'A'D' : \angle CAD$.

Case I. *When the plane* $\angle s$ $C'A'D'$ *and* CAD (Figs. 2 and 1), *are commensurable.*

Proof. Find a common measure of the $\angle s$ $C'A'D'$ and CAD, as $\angle CAK$, and let it be contained in $\angle C'A'D'$ n times, and in $\angle CAD$ m times.

Then $\qquad \angle C'A'D' : \angle CAD = n : m.$

Through $A'B'$ and the lines of division of $\angle C'A'D'$ pass planes, and through AB and the lines of division of $\angle CAD$ pass planes. These planes will divide the dihedral $\angle C'\text{–}A'B'\text{–}D'$ into n, and $\angle C\text{–}AB\text{–}D$ into m parts, all equal. Art. 552.

$$\therefore \angle C'\text{–}A'B'\text{–}D' : \angle C\text{–}AB\text{–}D = n : m.$$

Hence $\angle C'\text{–}A'B'\text{–}D' : \angle C\text{–}AB\text{–}D = \angle C'A'D' : \angle CAD$. (Why?)

CASE II. *When the plane angles $C'A'D'$ and CAD* (Figs. 3 and 1) *are incommensurable.*

Proof. Divide the $\angle CAD$ into any number of equal parts, and apply one of these parts to the $\angle C'A'D'$. It will be contained a certain number of times with a remainder, as $\angle LA'D'$, less than the unit of measure.

Hence the $\angle s\ C'A'L$ and CAD are commensurable.

$\therefore \angle C'-A'B'-L : \angle C-AB-D = \angle C'A'L : \angle CAD$. Case I.

If now we let the unit of measure be indefinitely diminished, the $\angle LA'D'$, which is less than the unit of measure, will be indefinitely diminished.

$\therefore \angle C'A'L \doteq \angle C'A'D'$ as a limit, and

$\angle C'-A'B'-L \doteq \angle C'-A'B'-D'$ as a limit. Art. 251.

Hence $\dfrac{\angle C'-A'B'-L}{\angle C-AB-D}$ becomes a variable, with $\dfrac{\angle C'-A'B'-D'}{\angle C-AB-D}$ as its limit; Art. 253, 3.

Also $\dfrac{\angle C'A'L}{\angle CAD}$ becomes a variable with $\dfrac{\angle C'A'D'}{\angle CAD}$ as its limit. Art. 253, 3.

But the variable $\dfrac{\angle C'-A'B'-L}{\angle C-AB-D}$ = the variable $\dfrac{\angle C'A'L}{\angle CAD}$ always. Case I.

$\therefore$ the limit $\dfrac{\angle C'-A'B'-D'}{\angle C-AB-D}$ = the limit $\dfrac{\angle C'A'D'}{\angle CAD}$. Art. 254.

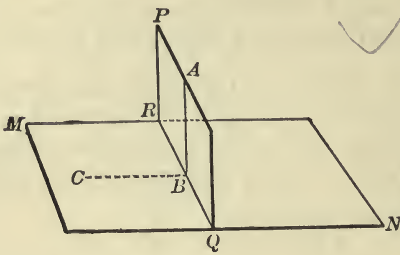
Q. E. D.

Ex. 1. How many straight lines are necessary to indicate a dihedral angle (as $\angle E-AB-D$, p. 338)? How many straight lines are necessary to indicate the plane angle of a dihedral angle? Hence, what is the advantage of using a plane angle of a dihedral angle instead of the dihedral angle itself?

Ex. 2. Give three additional properties of dihedral angles analogous to properties of plane angles given in Book I.

PROPOSITION XVII. THEOREM

555. *If a straight line is perpendicular to a plane, every plane drawn through that line is perpendicular to the plane.*



Given the line $AB \perp$ plane MN, and the plane PQ passing through AB and intersecting MN in RQ.

To prove $\quad PQ \perp MN.$

Proof. In the plane MN draw $BC \perp RQ$ at B.

But $\quad AB \perp RQ.$ Art. 505.

$\therefore \angle ABC$ is the plane $\angle$ of the dihedral $\angle P\text{-}RQ\text{-}M$. Art. 548.

But $\quad \angle ABC$ is a right $\angle$, Art. 505.
(*for* $AB \perp MN$ *by hyp.*).

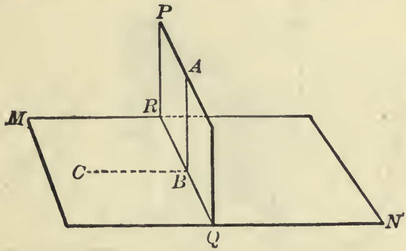
$\therefore PQ \perp MN.$ Art. 547.

Q. E. D.

556. COR. *A plane perpendicular to the edge of a dihedral angle is perpendicular to each of the two faces forming the dihedral angle.*

PROPOSITION XVIII. THEOREM

557. *If two planes are perpendicular to each other, a straight line drawn in one of them perpendicular to their line of intersection is perpendicular to the other plane.*



Given the plane $PQ \perp$ plane MN and intersecting it in the line RQ; and AB a line in $PQ \perp RQ$.

To prove $AB \perp$ plane MN.

Proof. In the plane MN draw $BC \perp RQ$.

$\therefore \angle ABC$ is the plane $\angle$ of the dihedral $\angle P-RQ-M$. Art. 548.

$\therefore \angle ABC$ is a rt. $\angle$, Art. 554.
(*for $P-RQ-M$ is a right dihedral $\angle$*).

$\therefore AB \perp BC$ and RQ at their intersection.

$\therefore AB \perp$ plane MN. (Why?)

Q. E. D.

558. COR. 1. *If two planes are perpendicular to each other, a perpendicular to one of them at any point of their intersection will lie in the other plane.*

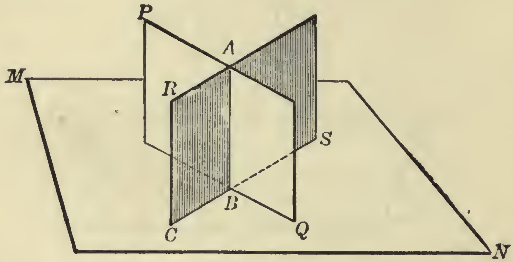
For, in the above figure, a $\perp$ erected at the point B in the plane MN must coincide with AB lying in the plane PQ and $\perp MN$, for at a given point in a plane only one $\perp$ can be drawn to that plane (Art. 515).

559. COR. 2. *If two planes are perpendicular to each other, a perpendicular to one plane, from a point in the other plane, will lie in the other plane.*

X

Proposition XIX. Theorem

560. *If two intersecting planes are each perpendicular to a third plane, their line of intersection is perpendicular to the third plane.*



Given the planes PQ and $RS \perp$ plane MN, and intersecting in the line AB.

To prove $AB \perp$ plane MN.

Proof. At the point B in which the three planes meet erect a $\perp$ to the plane MN. This $\perp$ must lie in the plane PQ, and also in the plane RS. Art. 558.

Hence this $\perp$ must coincide with AB, the intersection of PQ and RS. Art. 508, 2.

$\therefore AB \perp$ plane MN.

Q. E. D.

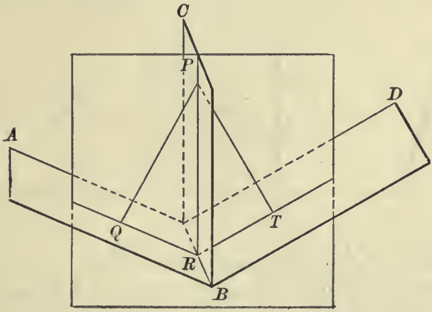
561. Cor. *If two planes, including a right dihedral angle, are each perpendicular to a third plane, the intersection of any two of the planes is perpendicular to the third plane, and each of the three lines of intersection is perpendicular to the other two.*

Ex. 1. Name all the dihedral angles on the above figure.

Ex. 2. If $\angle CBQ = 30°$, find the ratio of each pair of dihedral $\measuredangle$.

Proposition XX. Theorem

562. *Every point in the plane which bisects a given dihedral angle is equidistant from the faces of the dihedral angle.*



Given plane CB bisecting the dihedral $\angle A\text{-}BR\text{-}D$, P any point in plane BC, PQ and $PT \perp$ faces BA and BD, respectively.

To prove $PQ = PT$.

Proof. Through PQ and PT pass a plane intersecting AB in QR, BD in RT, and BC in PR.

Then plane $PQT \perp$ planes AB and BD. Art. 555.

$\therefore$ plane $PQT \perp$ line RB, the intersection of the planes AB and BD. Art. 560.

$\therefore RB \perp RQ$, RP and RT. Art. 505.

$\therefore$ $\measuredangle$ QRP and PRT are the plane $\measuredangle$ of the dihedral $\measuredangle$ $A\text{-}BR\text{-}P$ and $P\text{-}BR\text{-}D$. Art. 548.

But these dihedral $\measuredangle$ are equal. Hyp.

$\therefore \angle QRP = \angle PRT$. Art. 554.

$\therefore$ rt. $\triangle PQR =$ rt. $\triangle PRT$. (Why?)

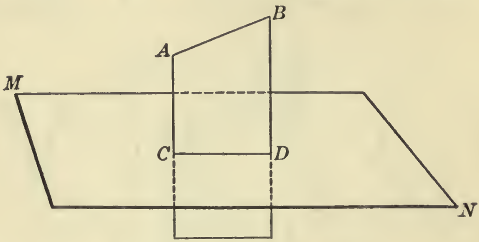
$\therefore PQ = PT$. (Why?)

Q. E. D.

563. *The locus of all points equidistant from the faces of a dihedral angle is the plane bisecting the dihedral angle.*

Proposition XXI. Problem

564. *Through any straight line not perpendicular to a given plane, to pass a plane perpendicular to the given plane.*



Given the line AB not $\perp$ plane MN.

To construct a plane passing through AB and $\perp$ MN.

Construction. From a point A in the line AB draw a $\perp$ AC to the plane MN. Art. 516.

Through the intersecting lines AB and AC pass the plane AD. Art. 503.

Then AD is the plane required.

Proof. The plane AD passes through AB. Constr.

Also plane $AD \perp$ plane MN, Art. 555.

(*for it contains AC, which is $\perp$ MN*).

Q. E. F.

565. Cor. 1. *Through a straight line not perpendicular to a given plane only one plane can be passed perpendicular to that plane.*

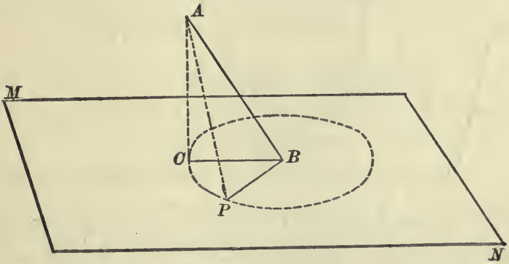
For, if two planes could be passed through $AB \perp$ plane MN, this intersection AB would be $\perp$ MN (Art. 560), which is contrary to the hypothesis.

566. Cor. 2. *The projection upon a plane of a straight line not perpendicular to that plane is a straight line.*

For, if a plane be passed through the given line $\perp$ to the given plane, the foot of a $\perp$ from any point in the line to the given plane will be in the intersection of the two planes (Art. 559).

Proposition XXII. Theorem

567. *The acute angle which a line makes with its projection on a plane is the least angle which it makes with any line of the plane through its foot.*



Given line AB meeting the plane MN in the point B, BC the projection of AB on MN, and PB any other line in the plane MN through B.

To prove that $\angle ABC$ is less than $\angle ABP$.

Proof. Lay off PB equal to CB, and draw AC and AP.

Then, in the $\triangle$ ABC and ABP,

$AB = AB$. (Why?)

$BC = BP$. (Why?)

But $AC < AP$. Art. 521.

$\therefore \angle ABC$ is less than $\angle ABP$ Art. 108.

Q. E. D.

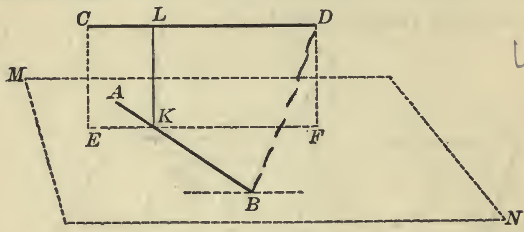
568. Def. **The inclination of a line to a plane** is the acute angle which the given line makes with its projection upon the given plane.

Ex. 1. A plane has an inclination of 47° to each of the faces of a dihedral angle and is parallel to the edge of the dihedral angle; how many degrees are in the plane angle of the dihedral angle?

Ex. 2. In the figure on page 345, if $PT=QT$, how large is the dihedral $\angle A\text{-}BR\text{-}D$? if $PT=RT$, how large is it?

PROPOSITION XXIII. PROBLEM

569. *To draw a common perpendicular to any two lines not in the same plane.*



Given the lines AB and CD not in the same plane.

To construct a line perpendicular to both AB and CD.

Construction. Through AB pass a plane $MN \parallel$ line CD. Art. 529.

Through CD pass a plane $CF \perp$ plane MN (Art. 564), and intersecting plane MN in the line EF.

Then $EF \parallel CD$ (Art. 528), $\therefore EF$ must intersect AB (which is not $\parallel CD$ by hyp.) in some point K.

At K in the plane CF draw $LK \perp EF$. Art. 274.

Then LK is the perpendicular required.

Proof. $LK \perp EF$. Constr.

$\therefore LK \perp CD$. Art. 123.

Also $LK \perp$ plane MN. Art. 557.

$\therefore LK \perp$ line AB. (Why?)

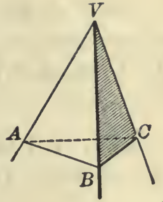
$\therefore LK \perp$ both CD and AB.

Q. E. F.

570. *Only one perpendicular can be drawn between two lines not in the same plane.*

For, if possible, in the above figure let another line BD be drawn $\perp AB$ and CD. Then, if a line be drawn through $B \parallel CD$, $BD \perp$ this line (Art. 123), and $\therefore \perp$ plane MN (Art. 509). Draw $DF \perp$ line EF; then $DF \perp$ plane MN (Art. 557). Hence from the point D two $\perp$s, DB and DF, are drawn to the plane MN, which is impossible (Art. 517).

POLYHEDRAL ANGLES



571. A **polyhedral angle** is the amount of opening between three or more planes meeting at a point.

Such an angle may be regarded as a portion of space cut out by the planes forming the angle.

572. The **vertex** of a polyhedral angle is the point in which the planes forming the angle meet; the **edges** are the lines in which the planes intersect; the **faces** are the portions of the planes forming the polyhedral angle which are included between the edges; the **face angles** are the angles formed by the edges.

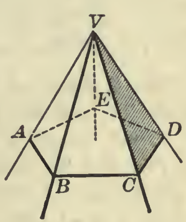
Each two adjacent faces of a polyhedral angle form a *dihedral angle.*

The **parts of a polyhedral angle** are its face angles and dihedral angles taken together.

573. Naming a polyhedral angle. A polyhedral angle is named either by naming the vertex, as V; or by naming the vertex and a point on each edge, as $V\text{–}ABC$.

In case two or more polyhedral angles have the same vertex, the latter method is necessary.

In the above polyhedral angle, the vertex is V; the edges are VA, VB, VC; the face angles are AVB, BVC, AVC.



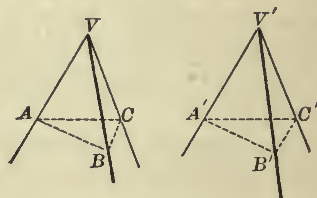
574. A **convex polyhedral angle** is a polyhedral angle in which a section made by a plane cutting all the edges is a convex polygon, as $V\text{–}ABCDE$.

575. A **trihedral angle** is a polyhedral angle having three faces; a **tetrahedral angle** is one having four faces, etc.

576. A trihedral angle is **rectangular, birectangular,** or **trirectangular,** according as it contains *one*, *two*, or *three* right dihedral angles.

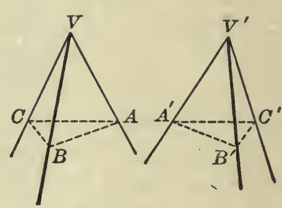
577. An **isosceles trihedral angle** is a trihedral angle two of whose face angles are equal.

578. Vertical polyhedral angles are polyhedral angles having the same vertex and the faces of one the faces of the other produced.

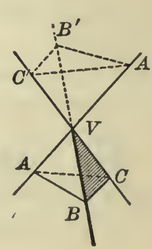


579. Two equal polyhedral angles are polyhedral angles having their corresponding parts equal and arranged in the same order, as $V\text{–}ABC$ and $V'\text{–}A'B'C'$.

Two equal polyhedral angles may be made to coincide.



580. Two symmetrical polyhedral angles are polyhedral angles having their corresponding parts equal but arranged in reverse order.



If the faces of a trihedral angle, $V\text{–}ABC$, be produced, they will form a vertical trihedral angle, $V\text{–}A'B'C'$, which is symmetrical to $V\text{–}ABC$. For, if $V\text{–}A'B'C'$ be rotated forward about a horizontal axis through V, the two trihedral angles are seen to have their corresponding parts equal but arranged in reverse order.

Similarly, any two vertical polyhedral angles are symmetrical.

581. Equivalence of symmetrical polyhedral angles. It has been shown in Plane Geometry (Art. 488) that two triangles (or polygons) symmetrical with respect to an axis have their corresponding parts equal and arranged in reverse order. By sliding two such figures about in a plane they cannot be made to coincide, but by lifting one of them up from the plane in which it lies and turning it over it may be made to coincide with the other figure.

Symmetrical polyhedral angles, however, cannot be made to coincide in any way; hence some indirect method of showing their equivalence is necessary. See Ex. 29, p. 358, and Arts. 789–792.

Ex. 1. Name the trihedral angles on the figure to Prop. XX. If $\angle PRQ = 90°$, what kind of trihedral angles are those on the figure? If $\angle PRQ = 30°$, what kind are they?

Ex. 2. Are two trirectangular trihedral angles necessarily equal? Prove this.

Ex. 3. Are two lines which are perpendicular to the same plane necessarily parallel? Are two planes which are perpendicular to the same plane necessarily parallel? Are two planes which are perpendicular to the same line necessarily parallel?

Ex. 4. Let the pupil cut out three pieces of pasteboard of the form indicated in the accompanying figures; cut them half through where the lines are dotted; fold them and fasten the edges so as to form three trihedral angles, two of which (Figs. 1 and 2) shall be equal and two (Figs. 1 and 3) symmetrical. By experiment, let the pupil find which pair may be made to coincide, and which not.

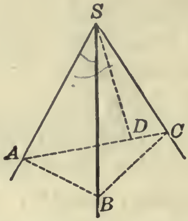
Fig. 1

Fig. 2

Fig. 3

Proposition XXIV. Theorem

582. *The sum of any two face angles of a trihedral angle is greater than the third face angle.*



Given the trihedral angle $S\text{-}ABC$, with angle ASC its greatest face angle.

To prove $\angle ASB + \angle BSC$ greater than $\angle ASC$.

Proof. In the face ASC draw SD, making $\angle ASD = \angle ASB$.

Take $SD = SB$.

In the face ASC draw the line ADC in any convenient direction, and draw AB and BC.

Then, in the △ ASB and ASD, $SA = SA$. (Why?)

$SB = SD$, and $\angle ASB = \angle ASD$. (Why?)

$\therefore \triangle ASB = \triangle ASD$. (Why?)

$\therefore AB = AD$. (Why?)

Also $AB + BC > AC$. (Why?)

Hence, subtracting the equals AB and AD.

$BC > DC$. (Why?)

Hence, in the △ BSC and DSC, $SC = SC$, $SB = SD$, and $BC > DC$. (Why?)

$\therefore \angle BSC$ is greater than $\angle DSC$. Art. 108.

To each of these unequals add the equals $\angle ASB$ and $\angle ASD$.

$\therefore \angle ASB + \angle BSC$ is greater than $\angle ASC$. (Why?)

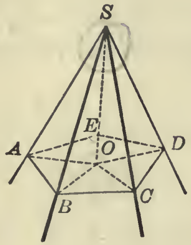
Q. E. D.

Ex. In the above figure, if $\angle ASC$ equals one of the other face angles at S, as $\angle ASB$, how is the theorem proved?

X

PROPOSITION XXV. THEOREM

583. *The sum of the face angles of any convex polyhedral angle is less than four right angles.*



Given the polyhedral angle $S\text{-}ABCDE$.

To prove the sum of the face $\measuredangle$ at S less than 4 rt. $\measuredangle$.

Proof. Pass a plane cutting the edges of the given polyhedral angle in the points A, B, C, D, E.

From any point O in the polygon $ABCDE$ draw OA, OB, OC, OD, OE.

Denote the $\triangle$ having the common vertex S as the S $\triangle$, and those having the common vertex O as the O $\triangle$.

Then the sum of the $\measuredangle$ of the S $\triangle$ = the sum of $\measuredangle$ of the O $\triangle$. Art. 134.

But $\angle SBA + \angle SBC$ is greater than $\angle ABC$,
$\angle SCB + \angle SCD$ is greater than $\angle BCD$, etc. } Art 582.

$\therefore$ the sum of the base $\measuredangle$ of the S $\triangle$ $>$ the sum of the base $\measuredangle$ of the O $\triangle$. Ax. 9.

$\therefore$ the sum of the vertex $\measuredangle$ of the S $\triangle$ $<$ the sum of the vertex $\measuredangle$ of the O $\triangle$, Ax. 11.

(*if unequals be subtracted from equals, the remainders are unequal in reverse order*).

But the sum of the $\measuredangle$ at O = 4 rt. $\measuredangle$. (Why?)

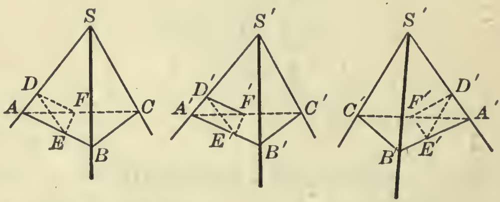
$\therefore$ the sum of face $\measuredangle$ at S $<$ 4 rt. $\measuredangle$. Ax. 8.

Q. E. D.

X

Proposition XXVI. Theorem

584. *If two trihedral angles have the three face angles of one equal to the three face angles of the other, the trihedral angles have their corresponding dihedral angles equal, and are either equal or symmetrical, according as their corresponding face angles are arranged in the same or in reverse order.*



Given the trihedral ∡ $S\text{–}ABC$ and $S'\text{–}A'B'C'$, having the face ∡ ASB, ASC and BSC equal to the face ∡ $A'S'B'$, $A'S'C'$ and $B'S'C'$, respectively.

To prove that the corresponding dihedral ∡ of $S\text{–}ABC$ and $S'\text{–}A'B'C'$ are equal, and that ∡ $S\text{–}ABC$ and $S'\text{–}A'B'C'$ are either equal or symmetrical.

Proof. On the edges of the trihedral ∡ take SA, SB, SC, $S'A'$, $S'B'$, $S'C'$ all equal.

Draw AB, AC, BC, $A'B'$, $A'C'$, $B'C'$.

Then, 1. In the ⟁ ASB and $A'S'B'$, $SA=S'A'$, $SB=S'B'$, and $\angle ASB=\angle A'S'B'$. (Why?)

$\therefore \triangle ASB=\triangle A'S'B'$. (Why?)

$\therefore AB=A'B'$. (Why?)

2. In like manner $AC=A'C'$, and $BC=B'C'$.

$\therefore \triangle ABC=\triangle A'B'C'$. (Why?)

d - f - d   f - f - f

3. Take D a convenient point in SA, and draw DE in the face ASB, and DF in the face ASC, each $\perp$ SA.

DE and DF meet AB and AC in points E and F, respectively, *(for ∠s SAB and SAC are acute).*

Similarly, take $S'D' = SD$ and construct $\triangle D'E'F'$.

Then, in the rt. △s ADE and $A'D'E'$, $AD = A'D'$, and $\angle DAE = D'A'E'$. (Why?)

$\therefore \triangle ADE = \triangle A'D'E'$. (Why?)

$\therefore AE = A'E'$, and $DE = D'E'$. (Why?)

4. In like manner it may be shown that $AF = A'F'$, and $DF = D'F'$.

$\therefore \triangle AEF = \triangle A'E'F'$. (Why?)

And $EF = E'F'$ (Why?)

5. Hence, in the △s DEF and $D'E'F'$, $DE = D'E'$, $DF = D'F'$ and $EF = E'F'$. (Why?)

$\therefore \triangle DEF = \triangle D'E'F'$. (Why?)

$\therefore \angle EDF = \angle E'D'F'$. (Why?)

But these ∠s are the plane ∠s of the dihedral ∠s whose edges are SA and $S'A'$.

$\therefore$ dihedral $\angle B\text{-}AS\text{-}C =$ dihedral $\angle B'\text{-}A'S'\text{-}C'$ Art. 552.

In like manner it may be shown that the dihedral ∠s at SB and $S'B'$ are equal; and that those at SC and $S'C'$ are equal.

$\therefore$ the trihedral ∠s S and S' are either equal or symmetrical. Arts. 579, 580.

Q. E. D.

EXERCISES. GROUP 64

THEOREMS CONCERNING THE LINE AND PLANE IN SPACE

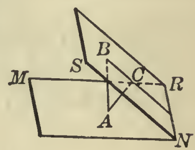
Ex. 1. A segment of a line not parallel to a plane is longer than its projection in the plane.

Ex. 2. Equal straight lines drawn from a point to a plane are equally inclined to the plane.

Ex. 3. A line and plane perpendicular to the same plane are parallel.

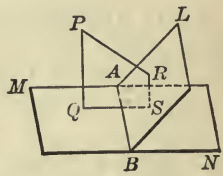
Ex. 4. If three planes intersecting in three straight lines are perpendicular to a plane, their lines of intersection are parallel.

Ex. 5. If a plane bisects any line at right angles, any point in the plane is equidistant from the ends of the line.



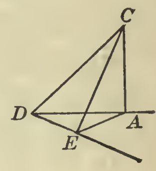
Ex. 6. Given $AB \perp$ plane MN,
and $AC \perp$ plane RS;
prove $BC \perp NR$.

Ex. 7. Given $PQ \perp$ plane MN,
$PR \perp$ plane BL,
and $RS \perp$ plane MN;
prove $QS \perp AB$.



Ex. 8. If a line is perpendicular to one of two intersecting planes, its projection on the other plane is perpendicular to the line of intersection of the two planes.

Ex. 9. Given $CE \perp DE$,
$AE \perp DE$,
and $\angle C\text{–}AD\text{–}E$ a rt. dihedral $\angle$;
prove $CA \perp$ plane DAE.



Ex. 10. The projections of two parallel lines on a plane are parallel. (Is the converse of this theorem also true?)

Ex. 11. If two parallel planes are cut by two non-parallel planes, the two lines of intersection in each of the parallel planes will make equal angles.

Ex. 12. If a line is perpendicular to a plane, any plane parallel to the line is perpendicular to the plane. (Is the converse true?)

Ex. 13. In the figure to Prop. VI, given $AB \perp MN$ and $AF \perp DC$; prove $BF \perp DC$.

Ex. 14. Two planes parallel to a third plane are parallel to each other.

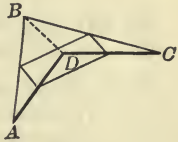
[SUG. Draw a line $\perp$ third plane.]

Ex. 15. The projections upon a plane of two equal and parallel straight lines are equal and parallel.

Ex. 16. A line parallel to two planes is parallel to their intersection.

Ex. 17. In the figure to Prop. XXII, if angle CBP is obtuse, prove the angle ABP obtuse.

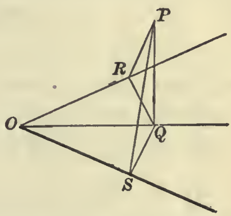
Ex. 18. In a quadrilateral in space (i. e., a quadrilateral whose vertices are not all in the same plane), show that the lines joining the midpoints of the sides form a parallelogram.



Ex. 19. The lines joining the midpoints of the opposite sides of a quadrilateral in space bisect each other.

Ex. 20. The planes bisecting the dihedral angles of a trihedral angle meet in a line every point of which is equidistant from the three faces.

[SUG. See Art. 562.]



Ex. 21. Given OQ bisecting $\angle ROS$,
$PQ \perp$ plane ROS,
$QR \perp OR$,
and $QS \perp OS$;
prove $PR = PS$, $PR \perp OR$,
and $PS \perp OS$.

Ex. 22. In a plane bisecting a given plane angle, and perpendicular to its plane, every point is equidistant from the sides of the angle.

[SUG. See Ex. 21; or through P any point in the bisecting plane pass planes $\perp$ to the sides of the $\angle$, etc.]

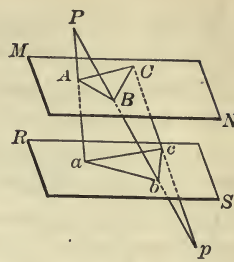
Ex. 23. In a trihedral angle, the three planes bisecting the three face angles at right angles to their respective planes, intersect in a line every point of which is equidistant from the three edges of the trihedral angle.

Ex. 24. If two face angles of a trihedral angle are equal, the dihedral angles opposite them are equal.

Ex. 25. In the figure to Prop. XXIV, prove that $\angle ASC + \angle BSC$ is greater than $\angle ASD + \angle BSD$.

Ex. 26. The common perpendicular to two lines in space is the shortest line between them.

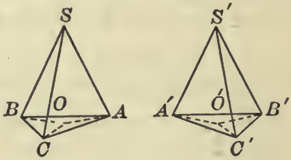
Ex. 27. Given $MN \parallel RS$,
and $PB = pb$;
prove $\angle ABC = \angle abc$,
and $\triangle\ ABC \approx \triangle\ abc$.



Ex. 28. Two isosceles symmetrical trihedral angles are equal.

Ex. 29. Any two symmetrical trihedral angles are equivalent.

[SUG. Take SA, SB, SC, $S'A'$, $S'B'$, $S'C'$, all equal. Pass planes ABC, $A'B'C'$. Draw SO and $S'O'$ ⊥ these planes. Then the trihedral ∠s are divided into three pairs of isosceles symmetrical trihedral ∠s, etc.]



EXERCISES. GROUP 65

LOCI IN SPACE

Find the locus of a point equidistant from

Ex. 1. Two parallel planes.

Ex. 2. Two given points.

Ex. 3. Three given points.

Ex. 4. Two intersecting lines.

Ex. 5. The three faces of a trihedral angle.

Ex. 6. The three edges of a trihedral angle.

Find the locus

Ex. 7. Of all lines passing through a given point and parallel to a given plane.

Ex. 8. Of all lines perpendicular to a given line at a given point in the line.

Ex. 9. Of all points in a given plane equidistant from a given point outside the plane.

Ex. 10. Of all points equidistant from two given points and from two parallel planes.

Ex. 11. Of all points equidistant from two given points and from two intersecting planes.

Ex. 12. Of all points at a given distance from a given plane and equidistant from two intersecting lines.

EXERCISES. GROUP 66

PROBLEMS CONCERNING THE POINT, LINE AND PLANE IN SPACE

Ex. 1. Through a given point pass a plane parallel to a given plane.

Ex. 2. Through a given point pass a plane perpendicular to a given plane.

Ex. 3. Through a given point to construct a plane parallel to two given lines which are not in the same plane.

Prove that only one plane can be constructed fulfilling the given conditions.

Ex. 4. Bisect a given dihedral angle.

Ex. 5. Draw a plane equally inclined to three lines which meet at a point.

Ex. 6. Through a given point draw a line parallel to two given intersecting planes.

Ex. 7. Find a point in a plane such that lines drawn to it from two given points without the plane make equal angles with the plane.

[Sug. See Ex. 23, p. 176.]

Ex. 8. Find a point in a given line equidistant from two given points.

Ex. 9. Find a point in a plane equidistant from three given points.

Ex. 10. Find a point equidistant from four given points not in a plane.

Ex. 11. Through a given point draw a line which shall intersect two given lines.

[Sug. Pass a plane through the given point and one of the given lines, and pass another plane through the given point and the other given line, etc.]

Ex. 12. Through a given point pass a plane cutting the edges of a tetrahedral angle so that the section shall be a parallelogram.

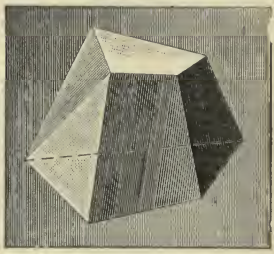
[Sug. Produce each pair of opposite faces to intersect in a straight line, etc.]

Book VII

POLYHEDRONS

585. A **polyhedron** is a solid bounded by planes.

586. The **faces** of a polyhedron are its bounding planes; the **edges** of a polyhedron are the lines of intersection of its faces.



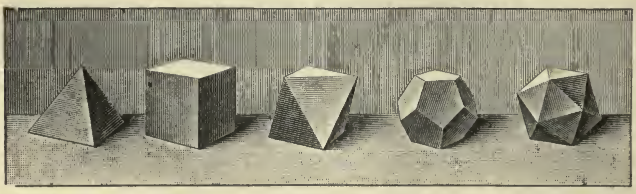
Polyhedron

A **diagonal** of a polyhedron is a straight line joining two of its vertices which are not in the same face. The **vertices** of a polyhedron are the points in which its edges meet or intersect.

587. A **convex polyhedron** is a polyhedron in which a section made by any plane is a convex polygon.

Only convex polyhedrons are to be considered in this book.

588. Classification of polyhedrons. Polyhedrons are sometimes classified according to the number of their faces. Thus, a **tetrahedron** is a polyhedron of four faces; a **hexahedron** is a polyhedron of six faces; an **octahedron** is one of eight, a **dodecahedron** one of twelve, and an **icosahedron** one of twenty faces.

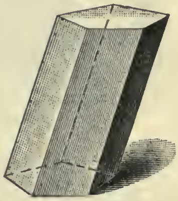


Tetrahedron Cube Octahedron Dodecahedron Icosahedron

The polyhedrons most important in practical life are those determined by their stability, the facility with which they can be made out of common materials, as wood and iron, the readiness with which they can be packed together, etc. Thus, prism means "something sawed off."

PRISMS AND PARALLELOPIPEDS

589. A **prism** is a polyhedron bounded by two parallel planes and a group of planes whose lines of intersection are parallel.



Prism

590. The **bases** of a prism are the faces formed by the two parallel planes; the **lateral faces** are the faces formed by the group of planes whose lines of intersection are parallel.

The **altitude** of a prism is the perpendicular distance between the planes of its bases.

The **lateral area** of a prism is the sum of the areas of the lateral faces.

591. Properties of a prism inferred immediately.

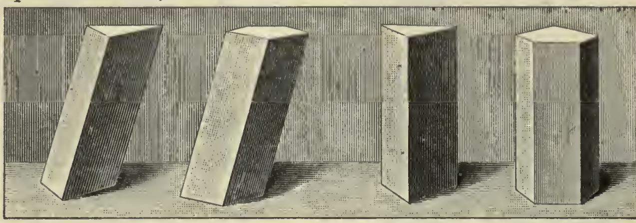
1. *The lateral edges of a prism are equal*, for they are parallel lines included between parallel planes (Art. 589) and are therefore equal (Art. 532).

2. *The lateral faces of a prism are parallelograms* (Art. 160), for their sides formed by the lateral edges are equal and parallel.

3. *The bases of a prism are equal polygons*, for their homologous sides are equal and parallel, each to each, (being opposite sides of a parallelogram), and their homologous angles are equal (Art. 538).

592. A **right section** of a prism is a section made by a plane perpendicular to the lateral edges.

593. A **triangular prism** is a prism whose base is a triangle; a **quadrangular prism** is one whose base is a quadrilateral, etc.



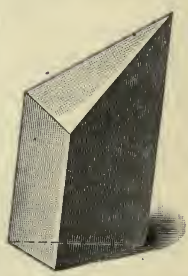
Oblique Prisms Right Prism Regular Prism

594. An **oblique prism** is a prism whose lateral edges are oblique to the bases.

595. A **right prism** is a prism whose lateral edges are perpendicular to the bases.

596. A **regular prism** is a right prism whose bases are regular polygons.

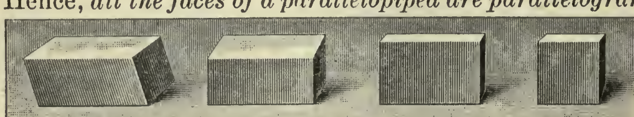
597. A **truncated prism** is that part of a prism included between a base and a section made by a plane ~~oblique~~ to the base and cutting all the lateral edges.



Truncated Prism

598. A **parallelopiped** is a prism whose bases are parallelograms.

Hence, *all the faces of a parallelopiped are parallelograms.*



Oblique Parallelopiped Right Parallelopiped Rectangular Parallelopiped Cube

599. A **right parallelopiped** is a parallelopiped whose lateral edges are perpendicular to the bases.

600. A **rectangular parallelopiped** is a right parallelopiped whose bases are rectangles.

Hence, *all the faces of a rectangular parallelopiped are rectangles.*

601. A **cube** is a rectangular parallelopiped whose edges are all equal.

Hence, *all the faces of a cube are squares.*

602. The **unit of volume** is a cube whose edge is equal to some linear unit, as a cubic inch, a cubic foot, etc. Imp.

603. The **volume of a solid** is the number of units of volume which the solid contains.

Being a *number*, a volume may often be determined from other numbers in certain expeditious ways, which it is one of the objects of geometry to determine.

604. **Equivalent solids** are solids whose volumes are equal.

Ex. 1. What is the least number of faces which a polyhedron can have?

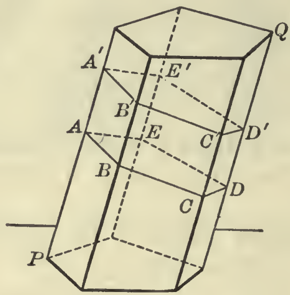
Ex. 2. A square right prism is what kind of a parallelopiped?

Ex. 3. Are there more right parallelopipeds or rectangular parallelopipeds? That is, which of these includes the other as a special case?

Ex. 4. Prove that if a given straight line is perpendicular to a given plane, and another straight line is perpendicular to another plane, and the two planes are parallel, then the two given lines are parallel.

Proposition I. Theorem

605. *Sections of a prism made by parallel planes cutting all the lateral edges are equal polygons.*



Given the prism PQ cut by $\parallel$ planes forming the sections AD and $A'D'$.

To prove section AD = section $A'D'$.

Proof. AB, BC, CD, etc., are $\parallel$ $A'B'$, $B'C'$, $C'D'$, etc., respectively. Art. 531.

$\therefore$ AB, BC, CD, etc., are equal to $A'B'$, $B'C'$, $C'D'$, etc., respectively. Art. 157.

Also $\angle\!\!s$ ABC, BCD, etc., are equal to $\angle\!\!s$ $A'B'C'$, $B'C'D'$, etc., respectively. Art. 538.

$\therefore$ $ABCDE = A'B'C'D'E'$, Art. 47.

(*for the polygons have all their parts equal, each to each, and* $\therefore$ *can be made to coincide*).

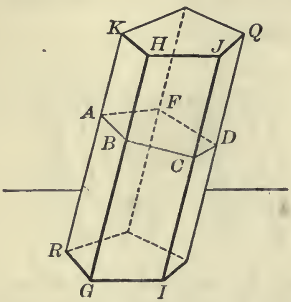
Q. E. D.

606. Cor. 1. *Every section of a prism made by a plane parallel to the base is equal to the base.*

607. Cor. 2. *All right sections of a prism are equal.*

Proposition II. Theorem

608. *The lateral area of a prism is equal to the product of the perimeter of a right section by a lateral edge.*



Given the prism RQ, with its lateral area denoted by S and lateral edge by E; and AD a right section of the given prism with its perimeter denoted by P.

To prove $S = P \times E$.

Proof. In the prism RQ, each lateral edge $= E$. Art. 591, 1.

Also $AB \perp GH$, $BC \perp IJ$, etc. Art. 505.

Hence
$$\left.\begin{aligned} \text{area} \ \square\ RH &= AB \times GH = AB \times E, \\ \text{area} \ \square\ GJ &= BC \times E, \\ \text{area} \ \square\ IQ &= CD \times E, \text{ etc.} \end{aligned}\right\} \quad \text{Art. 385.}$$

But S, the lateral area of the prism, equals the sum of the areas of the ▱s forming the lateral surface.

$\therefore$ adding, $S = (AB + BC + CD + \text{etc.}) \times E$. Ax. 2.

Or $S = P \times E$.

Q. E. D.

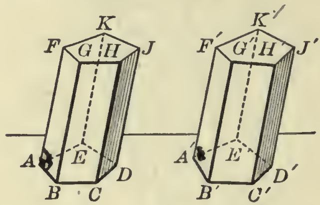
609. Cor. *The lateral area of a right prism equals the product of the perimeter of the base by the altitude.*

Ex. Find the lateral area of a right prism whose altitude is 12 in., and whose base is an equilateral triangle with a side of 6 in. Also find the total area of this figure.

Imp.

PROPOSITION III. THEOREM

610. *If two prisms have the three faces including a trihedral angle of one equal, respectively, to the three faces including a trihedral angle of the other, and similarly placed, the prisms are equal.*



Given the prisms AJ and $A'J'$, having the faces AK, AD, AG equal to the faces $A'K'$, $A'D'$, $A'G'$, respectively, and similarly placed.

To prove $AJ = A'J'$.

Proof. The face $\angle$s EAF, EAB and BAF are equal, respectively, to the face $\angle$s $E'A'F'$, $E'A'B'$ and $B'A'F'$. Hyp.

$\therefore$ trihedral $\angle A =$ trihedral $\angle A'$ Art. 584.

Apply the prism $A'J'$ to the prism AJ, making each of the faces of the trihedral $\angle A'$ coincide with corresponding equal face of the trihedral $\angle A$. Geom. Ax. 2.

$\therefore$ the plane $F'J'$ will coincide in position with the plane FJ, Art. 500.

(*for the points G', F', K' coincide with G, F, K, respectively*).

Also the point C' will coincide with the point C.

$\therefore$ $C'H'$ will take the direction of CH. Geom. Ax. 3.

$\therefore$ H' will coincide with H. Art. 508, 1.

In like manner J' will coincide with J.

Hence the prisms AJ and $A'J'$ coincide in all points.

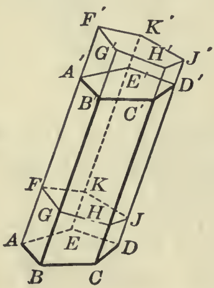
$\therefore$ $AJ = A'J'$. Art. 47.

611. COR. 1. *Two truncated prisms are equal if the three faces including a trihedral angle of one are equal to the three faces including a trihedral angle of the other.*

612. Cor. 2. *Two right prisms are equal if they have equal bases and equal altitudes.*

Proposition IV. Theorem

613. *An oblique prism is equivalent to a right prism whose base is a right section of the oblique prism and whose altitude is equal to a lateral edge of the oblique prism.*



Given the oblique prism AD', with the right section FJ; also the right prism FJ' whose lateral edges are each equal to a lateral edge of AD'.

To prove $AD' \approx FJ'$.

Proof. $AA' = FF'$. Hyp.

Subtracting FA' from each of these, $AF = A'F'$. (Why?)

Similarly $BG = B'G'$.

Also $AB = A'B'$, and $FG = F'G'$. Art. 155.

And $\angle$s of face AG = homologous $\angle$s of face $A'G'$. Art. 130.

$\therefore$ face AG = face $A'G'$, Art. 47.

(*for they have all their parts equal, each to each, and $\therefore$ can be made to coincide*).

In like manner face AK = face $A'K'$.

But face AD = face $A'D'$. Art. 591, 3.

$\therefore$ truncated prism AJ = truncated prism $A'J'$. Art. 611.

To each of these equals add the solid FD'.

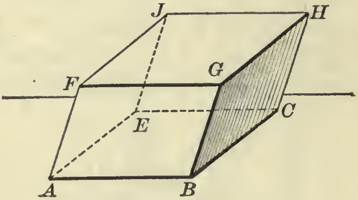
$\therefore AD' \approx FJ'$. (Why?)

Q. E. D.

Imp.

PROPOSITION V. THEOREM

614. *The opposite lateral faces of a parallelopiped are equal and parallel.*



Given the parallelopiped AH with the base AC.

To prove AG = and $\parallel$ EH, and AJ = and $\parallel$ BH.

Proof. The base AC is a $\square$. Art. 598.

$\therefore AB$ = and $\parallel$ EC. (Why?)

Also the lateral face AJ is a $\square$. Art. 591, 2.

$\therefore AF$ = and $\parallel$ EJ. (Why?)

$\therefore \angle BAF = \angle CEJ$. Art. 538.

And $\square AG = \square EH$. Art. 162.

Also plane $AG \parallel$ plane EH. Art. 538.

In like manner it may be shown that AJ and BH are equal and parallel.

Q. E. D.

615. COR. *Any two opposite faces of a parallelopiped may be taken as the bases.*

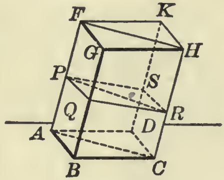
Ex. 1. How many edges has a parallelopiped? How many faces? How many dihedral angles? How many trihedral angles?

Ex. 2. Find the lateral area of a prism whose lateral edge is 10 and whose right section is a triangle whose sides are 6, 7, 8 in.

Ex. 3. Find the lateral area of a right prism whose lateral edge is 16 and whose base is a rhombus with diagonals of 6 and 8 in.

Proposition VI. Theorem

616. *A plane passed through two diagonally opposite edges of a parallelopiped divides the parallelopiped into two equivalent triangular prisms.*



Given the parallelopiped AH with a plane passed through the diagonally opposite edges AF and CH, forming the triangular prisms $ABC\text{–}G$ and $ADC\text{–}K$.

To prove $ABC\text{–}G \approx ADC\text{–}K$.

Proof. Construct a plane $\perp$ to one of the edges of the prism forming the right section $PQRS$, having the diagonal PR formed by the intersection of the plane $FHCA$.

Then $PQ \parallel SR$, and $QR \parallel PS$. (Why?)

$\therefore PQRS$ is a $\square$. (Why?)

$\therefore \triangle PQR = \triangle PSR$. (Why?)

But the triangular prism $ABC\text{–}G \approx$ a prism whose base is the right section PQR and whose altitude is AF. Art. 613.

Also the triangular prism $ADC\text{–}K \approx$ a prism whose base is the right section PSR and whose altitude is AF. (Why?)

But the prisms having the equal bases, PQR and PSR, and the same altitude, AF, are equal. Art. 612.

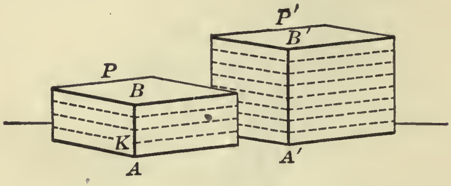
$\therefore ABC\text{–}G \approx ADC\text{–}K$. Ax. 1.

Q. E. D.

x

Proposition VII. Theorem

617. *If two rectangular parallelopipeds have equal bases, they are to each other as their altitudes.*



Given the rectangular parallelopipeds P' and P having equal bases and the altitudes $A'B'$ and AB.

To prove $P' : P = A'B' : AB$.

Case I. *When the altitudes $A'B'$ and AB are commensurable.*

Proof. Find a common measure of $A'B'$ and AB, as AK, and let it be contained in $A'B'$ n times and in AB m times.

Then $A'B' : AB = n : m$.

Through the points of division of $A'B'$ and AB pass planes parallel to the bases.

These planes will divide P' into n, and P into m small rectangular parallelopipeds, all equal. Art. 612.

$\therefore P' : P = n : m$.

$\therefore P' : P = A'B' : AB$. (Why?)

Case II. *When the altitudes $A'B'$ and AB are incommensurable.*

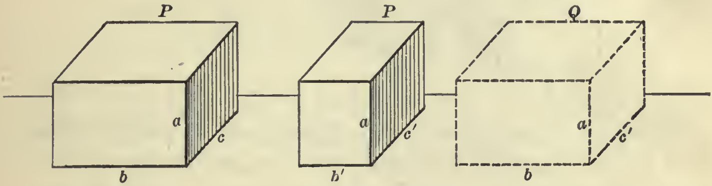
Let the pupil supply the proof, using the method of limits. (See Art. 554).

618. Def. The **dimensions of a rectangular parallelopiped** are the three edges which meet at one vertex.

619. Cor. *If two rectangular parallelopipeds have two dimensions in common, they are to each other as their third dimensions.*

Proposition VIII. Theorem

620. *Two rectangular parallelopipeds having equal altitudes are to each other as their bases.*



Given the rectangular parallelopipeds P and P' having the common altitude a, and the dimensions of their bases b, c and b', c', respectively.

To prove $$\frac{P}{P'}=\frac{b\times c}{b'\times c'}.$$

Proof. Construct the rectangular parallelopiped, Q, whose altitude is a and the dimensions of whose base are b and c'.

Then $$\frac{P}{Q}=\frac{c}{c'}.$$ Art. 619.

Also $$\frac{Q}{P'}=\frac{b}{b'}.$$ (Why?)

Multiplying the corresponding members of these equalities,

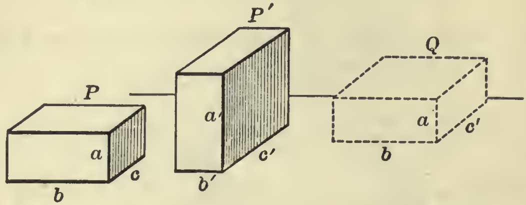
$$\frac{P}{P'}=\frac{b\times c}{b'\times c'}.$$ Ax. 4.

Q. E. D.

621. Cor. *Two rectangular parallelopipeds having one dimension in common are to each other as the products of the other two dimensions.*

PROPOSITION IX. THEOREM

622. *Any two rectangular parallelopipeds are to each other as the products of their three dimensions.*



Given the rectangular parallelopipeds P and P' having the dimensions a, b, c and a', b', c', respectively.

To prove $$\frac{P}{P'}=\frac{a \times b \times c}{a' \times b' \times c'}.$$

Proof. Construct the rectangular parallelopiped Q having the dimensions a, b, c'.

Then $$\frac{P}{Q}=\frac{c}{c'}.$$ Art. 619.

Also $$\frac{Q}{P'}=\frac{a \times b}{a' \times b'}.$$ Art. 621.

Multiplying the corresponding members of these equalities,

$$\frac{P}{P'}=\frac{a \times b \times c}{a' \times b' \times c'}.$$ Ax. 4.

Q. E. D.

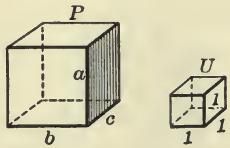
Ex. 1. Find the ratio of the volumes of two rectangular parallelopipeds whose edges are 5, 6, 7 in. and 7, 8, 9 in.

Ex. 2. Which will hold more, a bin 10 x 2 x 7 ft., or one 8 x 4 x 5 ft.?

Ex. 3. How many bricks 8 x 4 x 2 in. are necessary to build a wall 80 x 6 ft. x 8 in.?

Proposition X. Theorem

623. *The volume of a rectangular parallelopiped is equal to the product of its three dimensions.*



Given the rectangular parallelopiped P having the three dimensions a, b, c.

To prove volume of $P = a \times b \times c$.

Proof. Take as the unit of volume the cube U, whose edge is a linear unit.

Then $$\frac{P}{U} = \frac{a \times b \times c}{1 \times 1 \times 1}.$$ Art. 622.

The volume of P is the number of times P contains the unit of volume U, or $\frac{P}{U}$. Art. 603.

$\therefore$ volume of $P = a \times b \times c$.

Q. E. D.

For significance of this result, see Art. 2.

624. Cor. 1. *The volume of a cube is the cube of its edge.*

625. Cor. 2. *The volume of a rectangular parallelopiped is equal to the product of its base by its altitude.*

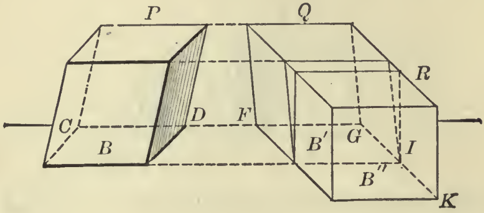
Ex. 1. Find the number of cubic inches in the volume of a cube whose edge is 1 ft. 3 in. How many bushels does this box contain, if 1 bushel = 2150.42 cu. in.?

Ex. 2. The measurement of the volume of a cube reduces to the measurement of the length of what single straight line?

PROPOSITION XI. THEOREM

Imp.

626. *The volume of any parallelopiped is equal to the product of its base by its altitude.*



Given the oblique parallelopiped P, with its base denoted by B, and its altitude by H.

To prove volume of $P = B \times H$.

Proof. In P produce the edge CD and all the edges parallel to CD.

On CD produced take $FG = CD$.

Pass planes through F and $G \perp$ the produced edges, forming the parallelopiped Q, with the rectangular base denoted by B'.

Similarly produce the edge GI and all the edges $\parallel GI$.

Take $IK = GI$, and pass planes through I and $K \perp$ the edges last produced, forming the rectangular parallelopiped R, with its base denoted by B''.

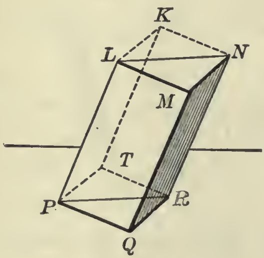
Then	$P \approx Q \approx R$.	Art. 613.
Also	$B \approx B' = B''$.	Art. 386.
But	volume of $R = B'' \times H$.	Art. 625.
	$\therefore$ volume of $P = B'' \times H$.	Ax. 1.
Or	volume of $P = B \times H$.	Ax. 8.

Q. E. D.

[**Outline Proof.** $P \approx Q \approx R = B'' \times H = B \times H$.]

PROPOSITION XII. THEOREM

627. *The volume of a triangular prism is equal to the product of its base by its altitude.*



Given the triangular prism $PQR\text{-}M$, with its volume denoted by V, area of base by B, and altitude by H.

To prove $V = B \times H$.

Proof. Upon the edges PQ, QR, QM, construct the parallelopiped QK.

Hence $QK =$ twice $PQR\text{-}M$. Art. 616.

But volume of $QK =$ area $PQRT \times H$. Art. 626.

$= 2B \times H$. Ax. 8.

$\therefore$ twice volume $PQR\text{-}M = 2B \times H$. Ax. 1.

$\therefore$ volume $PQR\text{-}M = B \times H$. Ax. 5.

Q. E. D.

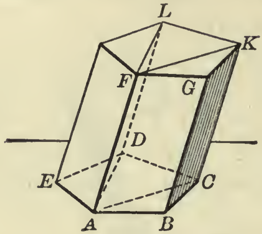
Ex. 1. If the altitude of a triangular prism is 18 in., and the base is a right triangle whose legs are 6 and 8 in., find the volume.

Ex. 2. Find the volume of a triangular prism whose altitude is 24, and the edges of whose base are 7, 8, 9. Also find the total surface.

Write out

PROPOSITION XIII. THEOREM

628. *The volume of any prism is equal to the product of its base by its altitude.*



Given the prism AK, with its volume denoted by V, area of base by B, and altitude by H.

To prove $V = B \times H$.

Proof. Through any lateral edge, as AF, and the diagonals of the base, AC and AD, drawn from its foot, pass planes.

These planes will divide the prism into triangular prisms.

Then V, the volume of the prism AK, equals the sum of the volumes of the triangular prisms. Ax. 6.

But the volume of each triangular prism = its base $\times H$.

Hence the sum of the volumes of the $\triangle$ prisms = the sum of the bases of the $\triangle$ prisms $\times H$.

$= B \times H$. Ax. 8.

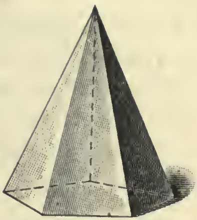
$\therefore V = B \times H$. Ax. 1.

Q. E. D.

629. COR. 1. *Two prisms are to each other as the products of their bases by their altitudes; prisms having equivalent bases and equal altitudes are equivalent.*

630. COR. 2. *Prisms having equivalent bases are to each other as their altitudes; prisms having equal altitudes are to each other as their bases.*

PYRAMIDS



Pyramid

631. A **pyramid** is a polyhedron bounded by a group of planes passing through a common point, and by another plane cutting all the planes of the group.

632. The **base** of a pyramid is the face formed by the cutting plane; the **lateral faces** are the faces formed by the group of planes passing through a common point; the **vertex** is the common point through which the group of planes passes; the **lateral edges** are the intersections of the lateral faces.

The **altitude** of a pyramid is the perpendicular from the vertex to the plane of the base.

The **lateral area** is the sum of the areas of the lateral faces.

633. Properties of pyramids inferred immediately.

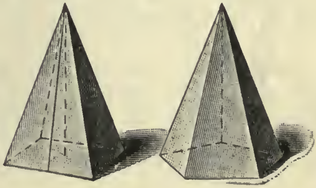
1. *The lateral faces of a pyramid are triangles* (Art. 508, 2).

2. *The base of a pyramid is a polygon* (Art. 508, 2).

634. A **triangular pyramid** is a pyramid whose base is a triangle; a **quadrangular pyramid** is a pyramid whose base is a quadrilateral, etc.

A triangular pyramid is also called a tetrahedron, for it has four faces. All these faces are triangles, and any one of them may be taken as the base.

635. A **regular pyramid** is a pyramid whose base is a regular polygon, and the foot of whose altitude coincides with the center of the base.



636. Properties of a regular pyramid inferred immediately.

1. *The lateral edges of a regular pyramid are equal*, for they are oblique lines drawn from a point to a plane cutting off equal distances from the foot of the perpendicular from the point to the plane (Art. 518).

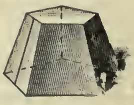
2. *The lateral faces of a regular pyramid are equal isosceles triangles.*

637. The **slant height of a regular pyramid** is the altitude of any one of its lateral faces.

The **axis** of a regular pyramid is its altitude.

638. A **truncated pyramid** is the portion of a pyramid included between the base and a section cutting all the lateral edges.

639. A **frustum of a pyramid** is the part of a pyramid included between the base and a plane parallel to the base.



The **altitude of a frustum of a pyramid** is the perpendicular distance between the planes of its bases.

640. Properties of a frustum of a pyramid inferred immediately.

1. *The lateral faces of a frustum of a pyramid are trapezoids.*

2. *The lateral faces of a frustum of a regular pyramid are equal isosceles trapezoids.*

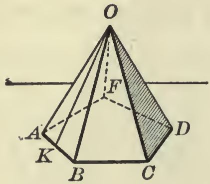
641. The **slant height of the frustum of a regular pyramid** is the altitude of one of its lateral faces.

Ex. 1. Show that the foot of the altitude of a regular pyramid coincides with the center of the circle circumscribed about the base.

Ex. 2. The perimeter of the midsection of the frustum of a pyramid equals one-half the sum of the perimeters of the bases.

Proposition XIV. Theorem

642. *The lateral area of a regular pyramid is equal to half the product of the slant height by the perimeter of the base.*



Given $O-ABCDF$ a regular pyramid with its lateral area denoted by S, slant height by L, and perimeter of its base by P.

To prove $S = \frac{1}{2} L \times P$.

Proof. The lateral faces OAB, OBC, etc., are equal isosceles △. Art. 636, 2.

Hence each lateral face has the same slant height, L.

∴ the area of each lateral face $= \frac{1}{2} L \times$ its base.

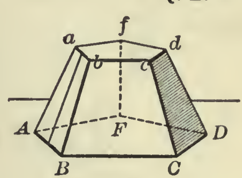
∴ the sum of all the lateral faces $= \frac{1}{2} L \times$ sum of bases.

$$= \frac{1}{2} L \times P.$$

$$\therefore S = \frac{1}{2} L \times P. \qquad \text{Ax. 8.}$$

Q. E. D.

643. Cor. *The lateral area of the frustum of a regular pyramid is equal to one-half the sum of the perimeters of its bases multiplied by its slant height.*



Ex. Find the lateral area of a regular square pyramid whose slant height is 32, and an edge of whose base is 16. Find the total area also.

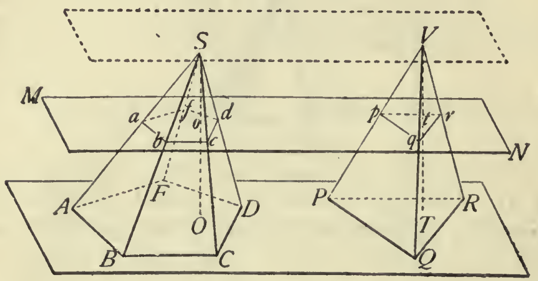
write out

Proposition XV. Theorem

644. *If a pyramid is cut by a plane parallel to the base,*

I. *The lateral edges and the altitude are divided proportionally;*

II. *The section is a polygon similar to the base.*



Given the pyramid $S\text{-}ABCDF$, with the altitude SO cut by a plane MN, which is parallel to the base and intersects the lateral edges in a, b, c, d, f and the altitude in o.

To prove I. $\dfrac{Sa}{SA}=\dfrac{Sb}{SB}=\dfrac{Sc}{SC}=\cdots=\dfrac{So}{SO}$.

II. The section $abcdf$ similar to the base $ABCDF$.

Proof. I. Pass a plane through the vertex $S \parallel MN$.

Then SA, SB, SC . . . SO, are lines intersected by three $\parallel$ planes.

$$\therefore \frac{Sa}{SA}=\frac{Sb}{SB}=\frac{Sc}{SC}=\cdots=\frac{So}{SO}. \qquad \text{Art. 539.}$$

II. $ab \parallel AB$. (Why?)

$\therefore \triangle$ Sab and SAB are similar. Art. 328.

In like manner the $\triangle$ Sbc, Scd, etc., are similar to the $\triangle$ SBC, SCD, etc., respectively.

$$\therefore \frac{ab}{AB}=\left(\frac{Sb}{SB}\right)=\frac{bc}{BC}=\left(\frac{Sc}{SC}\right)=\frac{cd}{CD}=\text{ etc.}$$

That is, the homologous sides of *abcdf* and *ABCDF* are proportional.

Also $\angle abc = \angle ABC$, $\angle bcd = \angle BCD$, etc. Art. 538.

$\therefore$ section *abcdf* is similar to the base *ABCDF*. Art. 321.

Q. E. D.

645. Cor. 1. *A section of a pyramid parallel to the base is to the base as the square of its distance from the vertex is to the square of the altitude of the pyramid.*

For $$\frac{abcdf}{ABCDF}=\frac{\overline{ab}^2}{\overline{AB}^2}.$$ (Why?)

But $$\frac{ab}{AB}=\frac{Sa}{SA}=\frac{So}{SO}\ \therefore\ \frac{\overline{ab}^2}{\overline{AB}^2}=\frac{\overline{So}^2}{\overline{SO}^2}.$$ (Why?)

$$\therefore\ \frac{abcdf}{ABCDF}=\frac{\overline{So}^2}{\overline{SO}^2}.$$ (Why?)

646. Cor. 2. *If two pyramids having equal altitudes are cut by a plane parallel to their bases at equal distances from the vertices, the sections have the same ratio as the bases.*

Let *S–ABCDF* and *V–PQR* be two pyramids cut as described.

Then $\frac{abcdf}{ABCDF}=\frac{\overline{So}^2}{\overline{SO}^2}$ (Art. 645); also $\frac{pqr}{PQR}=\frac{\overline{Vt}^2}{\overline{VT}^2}$. (Why?)

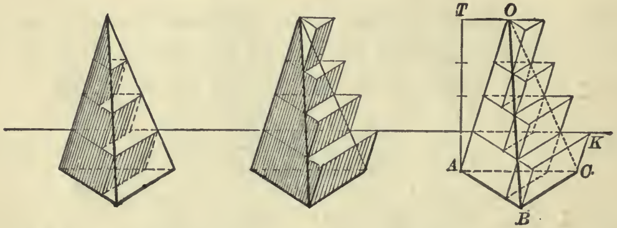
But $VT=SO$, and $Vt=So$. (Why?)

$$\therefore\ \frac{abcdf}{ABCDF}=\frac{pqr}{PQR},\ \text{or}\ \frac{abcdf}{pqr}=\frac{ABCDF}{PQR}.$$ (Why?)

647. Cor. 3. *If two pyramids have equal altitudes and equivalent bases, sections made by planes parallel to the bases at equal distances from the vertices are equivalent.*

PROPOSITION XVI. THEOREM

648. *The volume of a triangular pyramid is the limit of the sum of the volumes of a series of inscribed, or of a series of circumscribed prisms of equal altitude, if the number of prisms be indefinitely increased.*



Given the triangular prism $O\text{–}ABC$ with a series of inscribed, and also a series of circumscribed prisms, formed by passing planes which divide the altitude into equal parts, and by making the sections so formed first upper bases, then lower bases, of prisms limited by the next parallel plane.

To prove $O\text{–}ABC$ the limit of the sum of each series, if the number of prisms in each be indefinitely increased.

Proof. Each inscribed prism equals the circumscribed prism immediately above it. Art. 629.

$\therefore$ (sum of circumscribed prisms)—(sum of inscribed prisms) = lowest circumscribed prism, or $ABC\text{–}K$.

If the number of prisms be indefinitely increased, the altitude of each approaches zero as a limit.

Hence volume $ABC\text{–}K \doteq 0$, Art. 253, 2.
(for its base, ABC, is constant while its altitude $\doteq 0$).

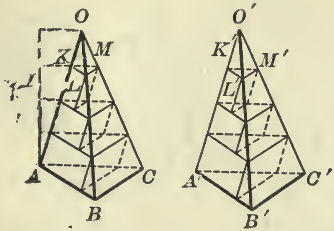
$\therefore$ (sum of circumscribed prisms)—(sum of inscribed prisms) $\doteq 0$.

$\therefore$ volume $O\text{–}ABC$—(either series of prisms) $\doteq 0$.
(for this difference < difference between the two series, which last difference $\doteq 0$).

$\therefore$ $O\text{–}ABC$ is the limit of the sum of the volumes of either series of prisms. Q. E. D.

Proposition XVII. Theorem

649. *If two triangular pyramids have equal altitudes and equivalent bases, they are equivalent.*



Given the triangular pyramids $O-ABC$ and $O'-A'B'C'$ having equivalent bases ABC and $A'B'C'$, and equal altitudes.

To prove $O-ABC \approx O'-A'B'C'$.

Proof. Place the pyramids so that they have the common altitude H, and divide H into any convenient number of equal parts.

Through the points of division and parallel to the plane of the bases of the pyramids, pass planes cutting the pyramids.

Using the sections so formed as upper bases, inscribe a series of prisms in each pyramid, and denote the volumes of the two series of prisms by V and V'.

The sections formed by each plane, as KLM and $K'L'M'$, are equivalent. Art. 647.

$\therefore$ each prism in $O-ABC \approx$ corresponding prism in $O'-A'B'C'$ (Art. 629). $\therefore V = V'$. Ax. 2.

Let the number of parts into which the altitude is divided be increased indefinitely.

Then V and V' become variables with $O-ABC$ and $O'-A'B'C'$ as their respective limits. Art. 648.

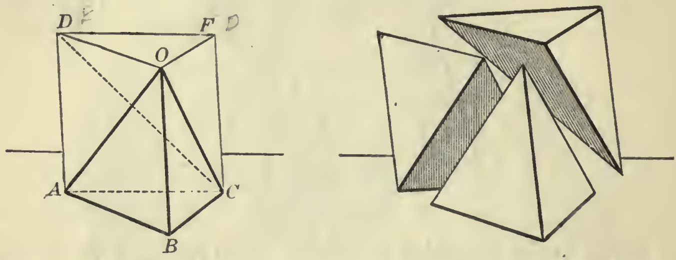
But $V \approx V'$ always. (Why?)

$\therefore O-ABC \approx O'-A'B'C'$. (Why?)

Q. E. D.

Proposition XVIII. Theorem

650. *The volume of a triangular pyramid is equal to one-third the product of its base by its altitude.*



Given the triangular pyramid $O\text{–}ABC$, having its volume denoted by V, the area of its base by B, and its altitude by H.

To prove $V = \frac{1}{3} B \times H$.

Proof. On ABC as a base, with OB as a lateral edge, construct the prism $ABC\text{–}DOF$.

Then this prism will be composed of the original pyramid $O\text{–}ABC$ and the quadrangular pyramid $O\text{–}ADFC$.

Through the edges OD and OC pass a plane intersecting the face $ADFC$ in the line DC, and dividing the quadrangular pyramid into the triangular pyramids $O\text{–}ADC$ and $O\text{–}DFC$.

Then $O\text{–}ADC \backsimeq O\text{–}DFC$. Art. 649.

(*for they have the common vertex O, and the equal bases ADC and DFC*).

But $O\text{–}DFC$ may be regarded as having C as its vertex and DOF as its base. Art. 634.

$\therefore O\text{–}DFC \backsimeq O\text{–}ABC$. Art. 649.

$\therefore$ the prism is made up of three equivalent pyramids.

$\therefore O\text{–}ABC = \frac{1}{3}$ the prism. Ax. 5.

But volume of prism $= B \times H$. Art. 627.

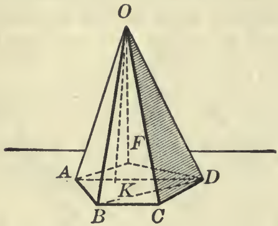
$\therefore O\text{–}ABC$, or $V = \frac{1}{3} B \times H$. Ax. 5.

Q. E. D.

Ex. Find the volume of a triangular pyramid whose altitude is 12 ft., and whose base is an equilateral triangle with a side of 15 ft.

Proposition XIX. Theorem

651. *The volume of any pyramid is equal to one-third the product of its base by its altitude.*



Given the pyramid $O\text{-}ABCDF$, having its volume denoted by V, the area of its base by B, and its altitude by H.

To prove $V = \frac{1}{3} B \times H$.

Proof. Through any lateral edge, as OD, and the diagonals of the base drawn from its foot, as AD and BD, pass planes dividing the pyramid into triangular pyramids.

Then V, the volume of the pyramid $O-ABCDF$, will equal the sum of the volumes of the triangular pyramids.

But the volume of each $\triangle$ pyramid $= \frac{1}{3}$ its base $\times H$. Art. 650.

Hence the sum of the volumes of $\triangle$ pyramids $= \frac{1}{3}$ sum of their bases $\times H$. Ax. 2.

$= \frac{1}{3} B \times H$. Ax. 8.

$\therefore V = \frac{1}{3} B \times H$. Ax. 1.

Q. E. D.

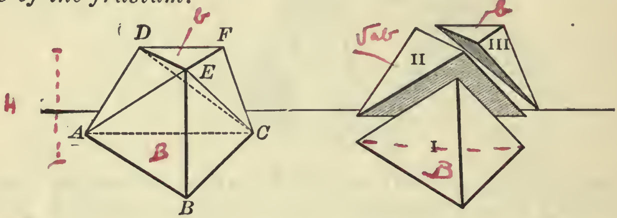
652. Cor. 1. *The volumes of two pyramids are to each other as the products of their bases and altitudes; pyramids having equivalent bases and equal altitudes are equivalent.*

653. Cor. 2. *Pyramids having equivalent bases are to each other as their altitudes; pyramids having equal altitudes are to each other as their bases.*

654. Scholium. *The volume of any polyhedron may be found by dividing the polyhedron into pyramids, finding the volume of each pyramid separately, and taking their sum.*

Proposition XX. Theorem

655. *The frustum of a triangular pyramid is equivalent to the sum of three pyramids whose common altitude is the altitude of the frustum, and whose bases are the lower base, the upper base, and a mean proportional between the two bases of the frustum.*



Given $ABC\text{–}DEF$ the frustum of a triangular pyramid, having the area of its lower base denoted by B, the area of its upper base by b, and its altitude by H.

To prove $ABC\text{–}DEF \approx$ three pyramids whose bases are B, b and $\sqrt{Bb}$, and whose common altitude is H.

Proof. Through E and AC, E and DC, pass planes dividing the frustum into three triangular pyramids. Then

1. $E\text{–}ABC$ has the base B and the altitude H.

2. $E\text{–}DFC$, that is, $C\text{–}DEF$, has the base b and the altitude H.

3. It remains to show that $E\text{–}ADC$ is equivalent to a pyramid having an altitude H, and a base that is a mean proportional between B and b.

Denoting the three pyramids by I, II, III,

$$\frac{\text{I}}{\text{II}}=\frac{\triangle ABE}{\triangle ADE}=\frac{AB}{DE}=\frac{AC}{DF}=\frac{\triangle ADC}{\triangle DFC}=\frac{\text{II}}{\text{III}}.$$

(Arts. 653, 391, 644, 321. Let the pupil supply the reason for each step in detail).

$$\therefore \frac{\text{I}}{\text{II}}=\frac{\text{II}}{\text{III}} \text{ (Ax. 1.) or } \text{II}=\sqrt{\text{I}\times\text{III}}. \qquad \text{Art. 303.}$$

$$\therefore E\text{–}ADC=\sqrt{(\tfrac{1}{3}H\times B)\,(\tfrac{1}{3}H\times b)}=\tfrac{1}{3}H\sqrt{B\times b}.$$

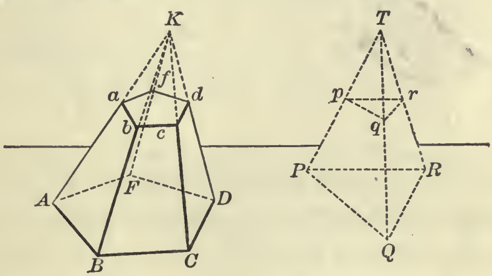
Hence, $ABC\text{–}DEF \approx$ sum of three pyramids, as described.

Q. E. D.

656. Formula for volume of frustum of a triangular pyramid. $V=\frac{1}{3}H(B+b+\sqrt{Bb})$.

Proposition XXI. Theorem

657. *The volume of the frustum of any pyramid is equivalent to the sum of the volumes of three pyramids, whose common altitude is the altitude of the frustum, and whose bases are the lower base. the upper base, and a mean proportional between the two bases of the frustum.*



omit
at presen

Given the frustum of a pyramid Ad, having its volume denoted by V, the area of its lower base by B, of its upper base by b, and its altitude by H.

To prove $V=\frac{1}{3}H(B+b+\sqrt{Bb})$.

Proof. Produce the lateral faces of Ad to meet in K.

Also construct a triangular pyramid with base PQR equivalent to $ABCDF$, and in the same plane with it, and with an altitude equal to the altitude of $K\text{-}ABCDF$. Produce the plane of ad to cut the second pyramid in pqr.

Then $pqr \approx abcdf$. Art. 647.

$\therefore$ pyramid $K\text{-}ABCDF \approx$ pyramid $T\text{-}PQR$. Art. 652.

Also pyramid $K\text{-}abcdf \approx$ pyramid $T\text{-}pqr$. (Why ?)

Subtracting, frustum $Ad \approx$ frustum Pr. Ax. 3.

But volume $Pr=\frac{1}{3}H(B+b+\sqrt{Bb})$.

$\therefore$ volume $Ad=\frac{1}{3}H(B+b+\sqrt{Bb})$. (Why ?)

Q. E. D.

Proposition XXII. Theorem

658. *A truncated triangular prism is equivalent to the sum of three pyramids, of which the base of the prism is the common base, and whose vertices are the three vertices of the inclined section.*

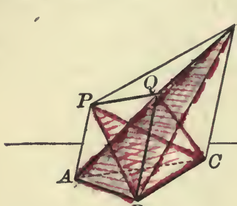


Fig. 1

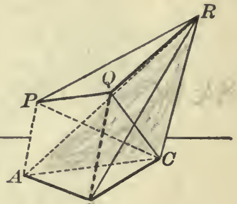


Fig. 2

Given the truncated triangular prism $ABC-PQR$.

To prove $ABC-PQR \approx$ the sum of the three pyramids $P-ABC$, $Q-ABC$ and $R-ABC$.

Proof. Pass planes through Q and AC, Q and PC, dividing the given figure into the three pyramids $Q-ABC$, $Q-APC$ and $Q-PRC$.

1. $Q-ABC$ has the required base and the required vertex Q.

2. $Q-APC \approx B-APC$, Art. 652.

(*for they have the same base, APC, and the same altitude, their vertices being in a line* $\parallel$ *base APC*).

But $B-APC$ may be regarded as having P for its vertex, and ABC for its base, as desired. Art. 634.

3. $Q-PRC \approx B-ARC$ (see Fig. 2). Art. 652.

(*for the base $ARC \approx$ base PRC* (Art. 390); *and the altitudes of the two pyramids are equal, the vertices Q and B being in line* $\parallel$ *plane $PACR$, in which the bases lie*).

But $B-ARC$ may be regarded as having R for its vertex, and ABC for its base, as desired. Art. 634.

$\therefore ABC\text{-}PQR \approx$ sum of three pyramids whose common base is ABC, and whose vertices are P, Q, R.

Q. E. D.

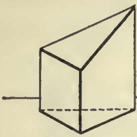


Fig. 3

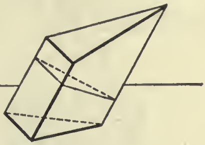


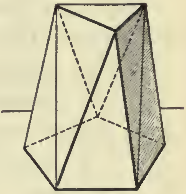
Fig. 4

659. Cor. 1. *The volume of a truncated right triangular prism* (Fig. 3) *is equal to the product of its base by one-third the sum of its lateral edges.*

660. Cor. 2. *The volume of any truncated triangular prism* (Fig. 4) *is equal to the product of the area of its right section by one-third the sum of its lateral edges.*

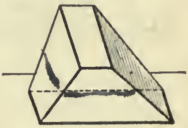
PRISMATOIDS

661. A **prismatoid** is a polyhedron bounded by two polygons in parallel planes, called **bases**, and by lateral faces which are either triangles, trapezoids or parallelograms.



Prismatoid

662. A **prismoid** is a prismatoid in which the bases have the same number of sides and have their corresponding sides parallel.



Prismoid

Ex. The volume of a truncated right parallelopiped equals the area of the lower base multiplied by one-fourth the sum of the lateral edges (or by a perpendicular from the center of the upper base to the lower base).

Proposition XXIII. Theorem

663. *The volume of a prismatoid is equal to one-sixth the product of its altitude by the sum of its bases and of four times the area of its midsection.*

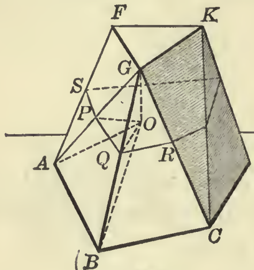


Fig. 1

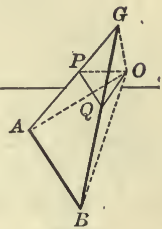


Fig. 2

Given the prismatoid $ABCD\text{–}FGK$, with bases B and b, midsection M, volume V, and altitude H.

To prove $V = \frac{1}{6} H\,(B + b + 4\,M)$.

Proof. Take any point O in the midsection, and through it and each edge of the prismatoid let planes be passed.

These planes will divide the figure into parts as follows:

1. A pyramid with vertex O, base $ABCD$ and altitude $\frac{1}{2}H$, and whose volume $\therefore = \frac{1}{6} H \times B$. Art. 651.

2. A pyramid with vertex O, base FGK, and altitude $\frac{1}{2} H$, and whose volume $\therefore = \frac{1}{6} H \times b$. (Why?)

3. Tetrahedrons like $O-ABG$ whose volume may be determined as follows (see Fig. 2):

$AB = 2\,PQ$. (Why?)

$\therefore \triangle AGB = 4 \triangle PGQ$. Art. 398.

$\therefore O\text{–}AGB = 4\,O\text{–}PGQ$. Art. 653.

But $O\text{–}PGQ$ (or $G\text{–}PQO$) $= \frac{1}{3} PQO \times \frac{1}{2} H = \frac{1}{6} H \times PQO$.

$\therefore O\text{–}AGB = \frac{1}{6} H \times 4 \triangle PQO$. (Why?)

$\therefore$ the sum of all tetrahedrons like $O\text{–}AGB = \frac{1}{6} H \times 4\,M$.

$\therefore V = \frac{1}{6} H \times B + \frac{1}{6} H \times b + \frac{1}{6} H \times 4\,M$.

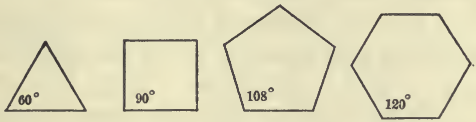
Or $V = \frac{1}{6} H\,(B + b + 4\,M)$. Q. E. D.

REGULAR POLYHEDRONS

664. Def. A **regular polyhedron** is a polyhedron all of whose faces are equal regular polygons, and all of whose polyhedral angles are equal. Thus, the cube is a regular polyhedron.

Proposition XXIV. Theorem

665. *But five regular polyhedrons are possible.*



Given regular polygons of 3, 4, 5, etc., sides.

To prove that regular polygons of the same number of sides can be joined to form polyhedral ∡ of a regular polyhedron in but five different ways, and that, consequently, but five regular polyhedrons are possible.

Proof. The sum of the face ∡ of any polyhedral angle $< 360°$. Art. 583.

1. Each ∠ of an *equilateral triangle* is 60°. Art. 134.

$3 \times 60°$, $4 \times 60°$ and $5 \times 60°$ are each less than 360°; but any larger multiple of $60° =$ or $> 360°$.

∴ but *three* regular polyhedrons can be formed with equilateral ▲ as faces.

2. Each ∠ of a *square* contains 90°. Art. 151.

$3 \times 90°$ is less than 360°, but any larger multiple of $90° =$ or $> 360°$.

∴ but *one* regular polyhedron can be formed with squares as faces.

3. Each ∠ of a regular *pentagon* is 108°. Art. 174.

$3 \times 108°$ is less than 360°, but any larger multiple of $108° > 360°$.

∴ but *one* regular polyhedron can be formed with regular pentagons as faces.

4. Each ∠ of a regular hexagon is 120°, and $3 \times 120° = 360°$.

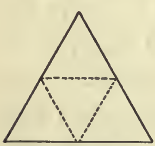
∴ no regular polyhedron can be formed with hexagons, or with polygons with a greater number of sides as faces.

∴ but five regular polyhedrons are possible.

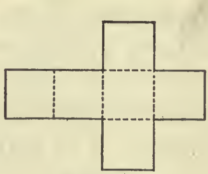
Q. E. D.

666. **The construction of the regular polyhedrons,** by the use of cardboard, may be effected as follows:

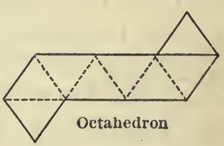
Draw on a piece of cardboard the diagrams given below. Cut the cardboard half through at the dotted lines and entirely through at the full lines. Bring the free edges together and keep them in their respective positions by some means, such as pasting strips of paper over them.



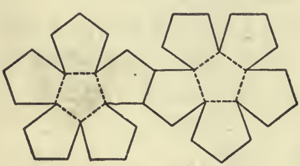
Tetrahedron



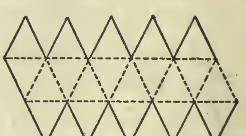
Hexahedron



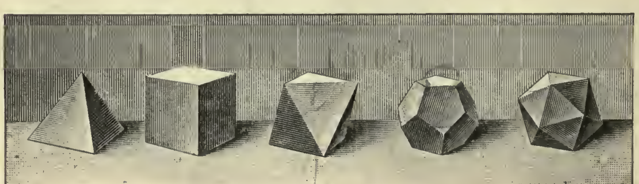
Octahedron



Dodecahedron



Icosahedron



POLYHEDRONS IN GENERAL

PROPOSITION XXV. THEOREM

667. *In any polyhedron, the number of edges increased by two equals the number of vertices increased by the number of faces.*

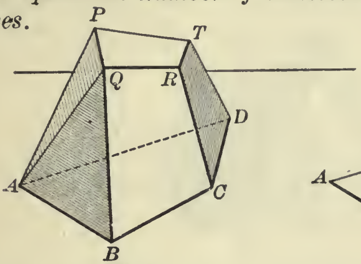


Fig. 1

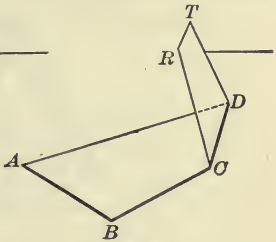


Fig. 2

Given the polyhedron AT, with the number of its vertices, edges and faces denoted by V, E and F, respectively.

To prove $E + 2 = V + F$.

Proof. Taking the single face $ABCD$, the number of edges equals the number of vertices, or $E = V$.

If another face, $CRTD$, be annexed (Fig. 2), *three* new *edges*, CR, RT, TD, are added and *two* new *vertices*, R and T.

$\therefore$ the number of edges gains one on the number of vertices, or $E = V + 1$.

If still another face, $BQRC$, be annexed, two new edges, BQ and QR, are added, and one new vertex, Q. $\therefore E = V + 2$.

With each new face that is annexed, the number of edges gains one on the number of vertices, till but one face is lacking.

The last face increases neither the number of edges nor of vertices.

Hence number of edges gains one on number of vertices, for every face except two, the first and the last, or gains $F - 2$ in all.

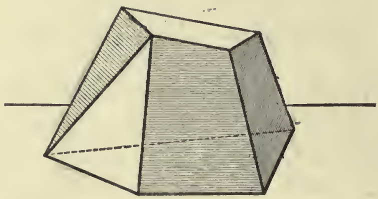
$\therefore$ for the entire figure, $E = V + F - 2$.

That is $E + 2 = V + F$. Ax. 2.

Q. E. D.

Proposition XXVI. Theorem

668. *The sum of the face angles of any polyhedron equals four right angles taken as many times, less two, as the polyhedron has vertices.*



Given any polyhedron, with the sum of its face angles denoted by S, and the number of its vertices, edges and faces denoted by V, E, F, respectively.

To prove $S = (V - 2)\ 4$ rt. $\angle$s.

Proof. Each edge of the polyhedron is the intersection of two faces, $\therefore$ the number of sides of the faces $= 2E$.

$\therefore$ the sum of the interior and exterior $\angle$s of the faces $= 2E \times 2$ rt. $\angle$s, or $E \times 4$ rt. $\angle$s. Art. 73.

But the sum of the exterior $\angle$s of each face $= 4$ rt. $\angle$s. Art. 175.

$\therefore$ the sum of exterior $\angle$s of the F faces $= F \times 4$ rt. $\angle$s. Ax. 4.

Subtracting the sum of the exterior $\angle$s from the sum of all the $\angle$s, the sum of the interior $\angle$s of the F faces $= (E \times 4$ rt. $\angle$s$) - (F \times 4$ rt. $\angle$s$)$.

Or $S = (E - F)\ 4$ rt. $\angle$s.

But $E + 2 = V + F$. Art. 667.

Hence $E - F = V - 2$. Ax. 3.

Substituting for $E - F$, $S = (V - 2)\ 4$ rt. $\angle$s. Ax. 8.

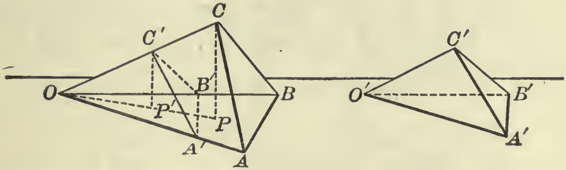
Q. E. D.

Ex. Verify the last two theorems in the case of the cube.

COMPARISON OF POLYHEDRONS. SIMILAR POLYHEDRONS

Proposition XXVII. Theorem

669. *If two tetrahedrons have a trihedral angle of one equal to a trihedral angle of the other, they are to each other as the products of the edges including the equal trihedral angles.*



Given the tetrahedrons $O\text{–}ABC$ and $O'\text{–}A'B'C'$, with their volumes denoted by V and V', respectively, and having the trihedral $\angle$s O and O' equal.

To prove $\dfrac{V}{V'}=\dfrac{OA \times OB \times OC}{O'A' \times O'B' \times O'C'}$.

Proof. Apply the tetrahedron $O'\text{–}A'B'C'$ to $O\text{–}ABC$ so that the trihedral $\angle O'$ shall coincide with its equal, the trihedral $\angle O$.

Draw CP and $C'P' \perp$ plane OAB, and draw OP the projection of OC in the plane OAB.

Taking OAB and $OA'B'$ as the bases, and CP and $C'P'$ as the altitudes of the pyramids $C\text{–}OAB$ and $C'\text{–}OA'B'$, respectively.

$$\frac{V}{V'}=\frac{\triangle OAB \times CP}{\triangle OA'B' \times C'P'}=\frac{\triangle OAB}{\triangle OA'B'}\times\frac{CP}{C'P'}. \qquad \text{Art. 652.}$$

But $$\frac{\triangle OAB}{\triangle OA'B'}=\frac{OA \times OB}{OA' \times OB'}. \qquad \text{Art. 397.}$$

In the similar rt. $\triangle$s OCP and $OC'P'$, $\dfrac{CP}{C'P'}=\dfrac{OC}{OC'}$. (Why?)

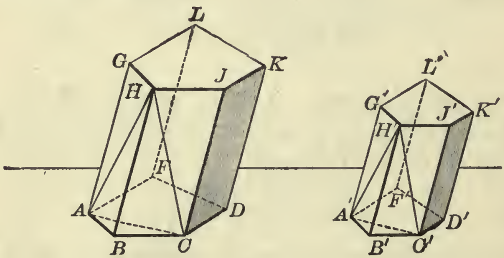
$$\therefore \frac{V}{V'}=\frac{OA \times OB \times OC}{OA' \times OB' \times OC'}=\frac{OA \times OB \times OC}{O'A' \times O'B' \times O'C'}. \qquad \text{Ax. 8.}$$

Q. E. D.

670. Def. **Similar polyhedrons** are polyhedrons having the same number of faces, similar, each to each, and similarly placed, and having their corresponding polyhedral angles equal.

Proposition XXVIII. Theorem

671. *Any two similar polyhedrons may be decomposed into the same number of tetrahedrons, similar, each to each, and similarly placed.*



Given P and P', two similar polyhedrons.

To prove that P and P' may be decomposed into the same number of tetrahedrons, similar, each to each.

Proof. Take H and H' any two homologous vertices of P and P'. Draw homologous diagonals in all the faces of P and P' except those faces which meet at H and H', separating the faces into corresponding similar triangles.

Through H and each face diagonal thus formed in P, and through H' and each face diagonal in P', pass planes.

Each corresponding pair of tetrahedrons thus formed may be proved similar.

Thus, in $H\text{-}ABC$ and $H'\text{-}A'B'C'$, the $\triangle$ HBA and $H'B'A'$ are similar. Art. 329.

In like manner $\triangle$ HBC and $H'B'C'$ are similar; and $\triangle$ ABC and $A'B'C'$ are similar.

Also $\frac{HA}{H'A'}=\left(\frac{HB}{H'B'}\right)=\frac{HC}{H'C'}=\left(\frac{BC}{B'C'}\right)=\frac{AC}{A'C'}$. Art. 321.

$\therefore$ △ AHC and $A'H'C'$ are similar. Art. 326.

Hence the corresponding faces of $H-ABC$ and $H'-A'B'C'$ are similar.

Also their homologous trihedral ∡ are equal. Art. 584.

$\therefore$ tetrahedron $H-ABC$ is similar to $H'-A'B'C'$. Art. 670.

After removing $H-ABC$ from P, and $H'-A'B'C'$ from P', the remaining polyhedrons are similar, for their faces are similar, and the remaining polyhedral ∡ are equal. Ax. 3.

By continuing this process, P and P' may be decomposed into the same number of tetrahedrons, similar, each to each, and similarly placed.

Q. E. D.

672. Cor. 1. *The homologous edges of similar polyhedrons are proportional;*

Any two homologous lines in two similar polyhedrons have the same ratio as any other two homologous lines.

673. Cor. 2. *Any two homologous faces of two similar polyhedrons are to each other as the squares of any two homologous edges or lines;*

The total areas of any two similar polyhedrons are to each other as the squares of any two homologous edges.

Ex. 1. In the figure, p. 395, if the edges meeting at O are 8, 9, 12 in., and those meeting at O' are 4, 6, 8 in., find the ratio of the volumes of the tetrahedrons.

Ex. 2. If the linear dimensions of one room are twice as great as the corresponding dimensions of another room, how will their surfaces (and $\therefore$ cost of papering) compare? How will their volumes compare?

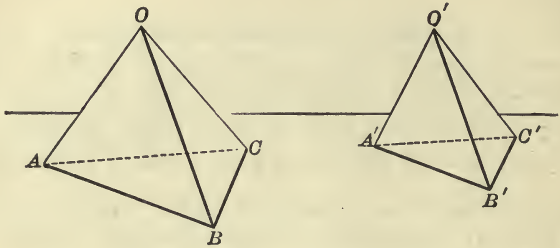
Ex. 3. How many 2 in. cubes can be cut from a 10 in. cube?

Ex. 4. If the bases of a prismoid are rectangles whose dimensions are a, b and b, a, and altitude is H, find the formula for the volume.

write out

Proposition XXIX. Theorem

674. *The volumes of two similar tetrahedrons are to each other as the cubes of any pair of homologous edges.*



Given the similar tetrahedrons $O\text{–}ABC$ and $O'\text{–}A'B'C'$.

To prove $$\frac{V}{V'}=\frac{\overline{OA}^3}{\overline{O'A'}^3}.$$

Proof. $$\frac{V}{V'}=\frac{OA\times OB\times OC}{O'A'\times O'B'\times O'C'}. \qquad \text{(Why?)}$$

$$=\frac{OA}{O'A'}\times\frac{OB}{O'B'}\times\frac{OC}{O'C'}.$$

But $$\frac{OA}{O'A'}=\frac{OB}{O'B'}=\frac{OC}{O'C'}. \qquad \text{Art. 672.}$$

$$\therefore \frac{V}{V'}=\frac{OA}{O'A'}\times\frac{OA}{O'A'}\times\frac{OA}{O'A'}=\frac{\overline{OA}^3}{\overline{O'A'}^3}. \qquad \text{Ax. 8.}$$

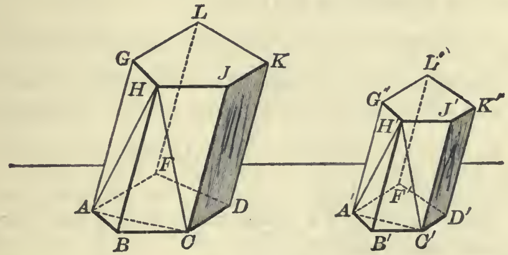
Ex. 1. In the above figures, if $AB=2\ A'B'$, find the ratio of V to V'. Find the same, if $AB=1\frac{1}{2}\ A'B'$.

Ex. 2 The measurement of the volume of a regular triangular prism reduces to the measurement of the lengths of how many straight lines? of a frustum of a regular square pyramid?

Ex. 3. Show how to construct out of pasteboard a regular prism, a parallelopiped, and a truncated square prism.

PROPOSITION XXX. THEOREM

675. *The volumes of any two similar polyhedrons are to each other as the cubes of any two homologous edges, or of any other two homologous lines.*



Given the polyhedrons AK and $A'K'$ having their volumes denoted by V and V', and HB and $H'B'$ any pair of homologous edges.

To prove $$\frac{V}{V'}=\frac{\overline{HB}^3}{\overline{H'B'}^3}.$$

Proof. Let the polyhedrons be decomposed into tetrahedrons, similar, each to each, and similarly placed. Art. 671.

Denote the volumes of the tetrahedrons in P by v_1, v_2, v_3 . . . and of those in P' by v'_1, v'_2, v'_3 . . .

Then $$\frac{v_1}{v'_1}=\frac{\overline{HB}^3}{\overline{H'B'}^3}.$$

Also $$\frac{v_1}{v'_1}=\frac{v_2}{v'_2}=\frac{v_3}{v'_3}=\cdots$$ Art. 674, Ax. 1.

(*for each of these ratios* $=\frac{\overline{HB}^3}{\overline{H'B'}^3}$.)

$$\therefore \frac{v_1+v_2+v_3+—}{v'_1+v'_2+v'_3+—}=\frac{v_1}{v'_1};\text{ that is, }\frac{V}{V'}=\frac{v_1}{v'_1}.$$ Art. 312.

$$\therefore \frac{V}{V'}=\frac{\overline{HB}^3}{\overline{H'B'}^3}.$$ Ax. 1.

Q. E. D.

EXERCISES. GROUP 67

THEOREMS CONCERNING POLYHEDRONS

Ex. 1. The lateral faces of a right prism are rectangles.

Ex. 2. A diagonal plane of a prism is parallel to every lateral edge of the prism not contained in the plane.

Ex. 3. The diagonals of a parallelopiped bisect each other.

Ex. 4. The square of a diagonal of a rectangular parallelopiped equals the sum of the squares of the three edges meeting at a vertex.

Ex. 5. Each lateral face of a prism is parallel to every lateral edge not contained in the face.

Ex. 6. Every section of a prism made by a plane parallel to a lateral edge is a parallelogram.

Ex. 7. If any two diagonal planes of a prism which are not parallel to each other are perpendicular to the base of the prism, the prism is a right prism.

Ex. 8. What part of the volume of a cube is the pyramid whose base is a face of the cube and whose vertex is the center of the cube?

Ex. 9. Any section of a regular square pyramid made by a plane through the axis is an isosceles triangle.

Ex. 10. In any regular tetrahedron, an altitude equals three times the perpendicular from its foot to any face.

Ex. 11. In any regular tetrahedron, an altitude equals the sum of the perpendiculars to the faces from any point within the tetrahedron.

Ex. 12. Find the simplest formula for the lateral area of a truncated regular prism of n sides.

Ex. 13. The sum of the squares of the four diagonals of a parallelopiped is equal to the sum of the squares of the twelve edges.

[SUG. Use Art. 352.]

Ex. 14. A parallelopiped is symmetrical with respect to what point?

Ex. 15. A rectanglar parallelopiped is symmetrical with respect to how many planes? (Let the pupil make a definition of a figure symmetrical with respect to a plane. See Arts. 486, 487.)

Ex. 16. The volume of a pyramid whose lateral edges are the three edges of the parallelopiped meeting at a point is what part of the volume of the parallelopiped?

Ex. 17. If a plane be passed through a vertex of a cube and the diagonal of a face not adjacent to the vertex, what part of the volume of the cube is contained by the pyramid so formed?

Ex. 18. If the angles at the vertex of a triangular pyramid are right angles and each lateral edge equals a, show that the volume of the pyramid is $\frac{a^3}{6}$.

Ex. 19. How large is a dihedral angle at the base of a regular pyramid, if the apothem of the base equals the altitude of the pyramid?

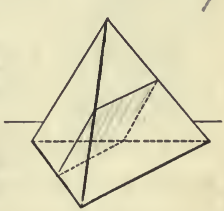
Ex. 20. The area of the base of a pyramid is less than the area of the lateral surface.

Ex. 21. The section of a triangular pyramid by a plane parallel to two opposite edges is a parallelogram.

If the pyramid is regular, what kind of a parallelogram does the section become?

Ex. 22. The altitude of a regular tetrahedron divides an altitude of the base into segments which are as $2:1$.

Ex. 23. If the edge of a regular tetrahedron is a, show that the slant height is $\frac{a\sqrt{3}}{2}$; and hence that the altitude is $\frac{a\sqrt{6}}{3}$, and the volume is $\frac{a^3\sqrt{2}}{12}$.



Ex. 24. If the midpoints of all the edges of a tetrahedron except two opposite edges be joined, a parallelogram is formed.

Ex. 25. Straight lines joining the midpoints of the opposite edges of a tetrahedron meet in a point and bisect each other.

Ex. 26. The midpoints of the edges of a regular tetrahedron are the vertices of a regular octahedron.

EXERCISES. GROUP 68

PROBLEMS CONCERNING POLYHEDRONS

Ex. 1. Bisect the volume of a given prism by a plane parallel to the base.

Ex. 2. Bisect the lateral surface of a given pyramid by a plane parallel to the base.

Ex. 3. Through a given point pass a plane which shall bisect the volume of a given parallelopiped.

Ex. 4. Given an edge, construct a regular tetrahedron.

Ex. 5. Given an edge, construct a regular octahedron.

Ex. 6. Pass a plane through the axis of a regular tetrahedron so that the section shall be an isosceles triangle.

Ex. 7. Pass a plane through a cube so that the section shall be a regular hexagon.

Ex. 8. Through three given lines no two of which are parallel pass planes which shall form a parallelopiped.

Ex. 9. From cardboard construct a regular square pyramid each of whose faces is an equilateral triangle.

EXERCISES. GROUP 69

REVIEW EXERCISES

Make a list of the properties of

Ex. 1. Straight lines in space.

Ex. 2. One line and one plane.

Ex. 3. Two or more lines and one plane.

Ex. 4. Two planes and one line.

Ex. 5. Two planes and two lines.

Ex. 6. Polyhedrons in general.

Ex. 7. Similar polyhedrons.

Ex. 8. Prisms in general.

Ex. 9. Right prisms.

Ex. 10. Parallelopipeds in general.

Ex. 11. Rectangular parallelopipeds.

Ex. 12. Pyramids in general.

Ex. 13. Regular pyramids.

Ex. 14. Frusta of pyramids.

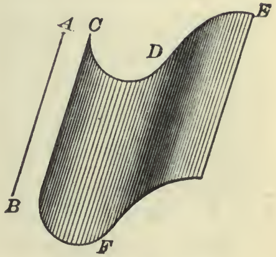
Ex. 15. Truncated prisms.

Book VIII

CYLINDERS AND CONES

CYLINDERS

676. A **cylindrical surface** is a curved surface generated by a straight line which moves so as constantly to touch a given fixed curve and constantly be parallel to a given fixed straight line.

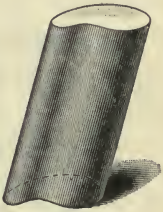


Cylindrical surface

Thus, every shadow cast by a point of light at a great distance, as by a star or the sun, approximates the cylindrical form, that is, is bounded by a cylindrical surface of light. Hence, in all radiations (as of light, heat, magnetism, etc.) from a point at a great distance, we are concerned with cylindrical surfaces and solids.

677. The **generatrix** of a cylindrical surface is the moving straight line; the **directrix** is the given curve, as *CDE*; an **element** of the cylindrical surface is the moving straight line in any one of its positions, as *DF*.

678. A **cylinder** is a solid bounded by a cylindrical surface and by two parallel planes.


Cylinder

The **bases** of a cylinder are its parallel plane faces; the **lateral surface** is the cylindrical surface included between the parallel planes forming its bases; the **altitude** of a cylinder is the distance between the bases.

The **elements** of a cylinder are the elements of the cylindrical surface bounding it.

679. Property of a cylinder inferred immediately. *All the elements of a cylinder are equal,* for they are parallel lines included between parallel planes (Arts. 532, 676).

The cylinders most important in practical life are those determined by their stability, the ease with which they can be made from common materials, etc.

Oblique circular cylinder

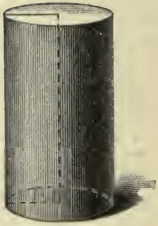
680. A **right cylinder** is a cylinder whose elements are perpendicular to the bases.

681. An **oblique cylinder** is one whose elements are oblique to the bases.

682. A **circular cylinder** is a cylinder whose bases are circles.

683. A **cylinder of revolution** is a cylinder generated by the revolution of a rectangle about one of its sides as an axis.

Hence, a cylinder of revolution is a **right circular cylinder.**



Cylinder of revolution

Some of the properties of this solid are derived most readily by considering it as generated by a revolving rectangle; and others, by regarding it as a particular kind of cylinder derived from the general definition.

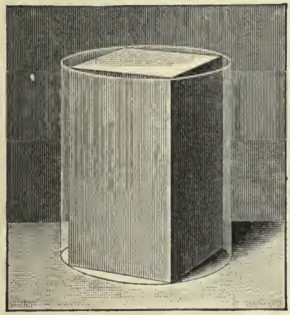
684. Similar cylinders of revolution are cylinders generated by similar rectangles revolving about homologous sides.

685. A **tangent plane** to a cylinder is a plane which contains one element of the cylinder, and which does not cut the cylinder on being produced.

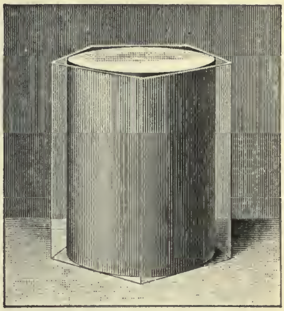
Ex. 1. A plane passing through a tangent to the base of a circular cylinder and the element drawn through the point of contact is tangent to the cylinder. (For if it is not, etc.)

Ex. 2. If a plane is tangent to a circular cylinder, its intersection with the plane of the base is tangent to the base.

686. A prism inscribed in a cylinder is a prism whose lateral edges are elements of the cylinder, and whose bases are polygons inscribed in the bases of the cylinder.



Inscribed prism



Circumscribed prism

687. A prism circumscribed about a cylinder is a prism whose lateral faces are tangent to the cylinder, and whose bases are polygons circumscribed about the bases of the cylinder.

688. A section of a cylinder is the figure formed by the intersection of the cylinder by a plane.

A **right section** of a cylinder is a section formed by a plane perpendicular to the elements of the cylinder.

689. Properties of circular cylinders. By Art. 441 the area of a circle is the limit of the area of an inscribed or circumscribed polygon, and the circumference is the limit of the perimeters of these polygons; hence

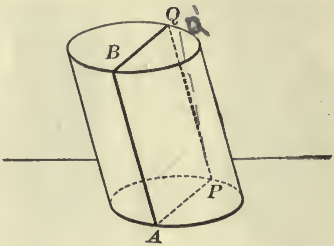
1. *The volume of a circular cylinder is the limit of the volume of an inscribed or circumscribed prism.*

2. *The lateral area of a circular cylinder is the limit of the lateral area of an inscribed or circumscribed prism.*

Also, 3. *By methods too advanced for this book, it may be proved that the perimeter of a right section is the limit of the perimeter of a right section of an inscribed or circumscribed prism.*

Proposition I. Theorem

690. *Every section of a cylinder made by a plane passing through an element is a parallelogram.*



Given the cylinder AQ cut by a plane passing through the element AB and forming the section $ABQP$.

To prove $ABQP$ a $\square$.

Proof. $AP \parallel BQ$. Art. 531.

It remains to prove that PQ is a ~~straight~~ line $\parallel AB$.

Through P draw a line PQ in the cutting plane $\parallel AB$. ?

be an element

This line will also lie in the cylindrical surface. Art. 676.

being in both surfaces, it must be its intersection

$\therefore$ this line must coincide with PQ,

(*for the line drawn lies in both the cutting plane and the cylindrical surface, hence, it must be their intersection*).

$\therefore PQ$ is a ~~straight~~ line $\parallel AB$.

$\therefore ABQP$ is a $\square$. (Why?)

Q. E. D.

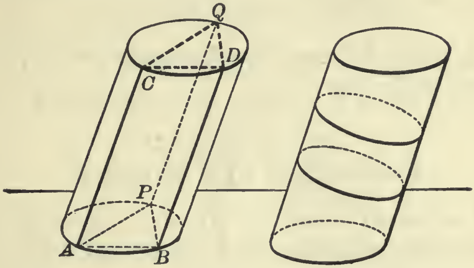
691. Cor. *Every section of a right cylinder made by a plane passing through an element is a rectangle.*

Ex. 1. A door swinging on its hinges generates what kind of a solid?

Ex. 2. Every section of a parallelopiped made by a plane intersecting all its lateral edges is a parallelogram.

Proposition II. Theorem

692. *The bases of a cylinder are equal.*



Given the cylinder AQ with the bases APB and CQD.

To prove base APB = base CQD.

Proof. Let AC and BD be any two fixed elements in the surface of the cylinder AQ.

Take P, any point except A and B in the perimeter of the base, and through it draw the element PQ.

Draw AB, AP, PB, CD, CQ, QD.

Then AC and BD are $=$ and $\|$. (Why?)

$\therefore AD$ is a ▱. (Why?)

Similarly AQ and BQ are ▱s.

$\therefore AB = CD$, $AP = CQ$, and $BP = DQ$. (Why?)

$\therefore \triangle APB = \triangle CQD$. (Why?)

Apply the base APB to the base CQD so that AB coincides with CD. Then P will coincide with Q,

(*for* $\triangle APB = \triangle CQD$).

But P is any point in the perimeter of the base APB.

$\therefore$ every point in the perimeter of the lower base will coincide with a corresponding point of the perimeter of the upper base.

$\therefore$ the bases will coincide and are equal. Art. 47.

Q. E. D.

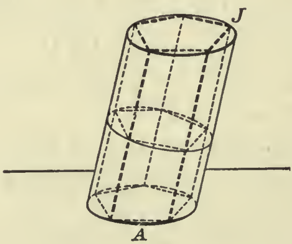
693. Cor. 1. *The sections of a cylinder made by two parallel planes cutting all the elements are equal.*

For the sections thus formed are the bases of the cylinder included between the cutting planes.

694. Cor. 2. *Any section of a cylinder parallel to the base is equal to the base.*

Proposition III. Theorem

695. *The lateral area of a circular cylinder is equal to the product of the perimeter of a right section of the cylinder by an element.*



Given the circular cylinder AJ, having its lateral area denoted by S, an element by E, and the perimeter of a right section by P.

To prove $S = P \times E$.

Proof. Let a prism with a regular polygon for its base be inscribed in the cylinder.

Denote the lateral area of the inscribed prism by S', and the perimeter of its right section by P'.

Then the lateral edge of the inscribed prism is an element of the cylinder. Constr

$\therefore\ S' = P' \times E.$ Art. 608.

If the number of lateral faces of the inscribed prism be indefinitely increased,

S' will approach S as a limit. Art. 689, 2.

P' will approach P as a limit. Art. 689, 3.

And $P' \times E$ will approach $P \times E$ as a limit. Art. 253, 2.

But $S' = P' \times E$ always. (Why?)

$\therefore\ S = P \times E.$ (Why?)

Q. E. D.

696. Cor. 1. *The lateral area of a cylinder of revolution is equal to the product of the circumference of its base by its altitude.*

697. Formulas for lateral area and total area of a cylinder of revolution. Denoting the lateral area of a cylinder of revolution by S, the total area by T, the radius by R, and the altitude by H.

$$S = 2\,\pi RH.$$

$$T = 2\,\pi RH + 2\,\pi R^2 \ \therefore\ T = 2\,\pi R\,(H + R).$$

Ex. 1. If, in a cylinder of revolution, $H = 10$ in. and $R = 7$ in., find S and T.

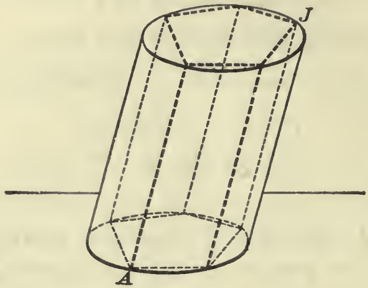
Ex. 2. If the altitude of a cylinder of revolution equals the radius of the base ($H = R$), what do the formulas for S and T become in terms of R? also, in terms of H?

Ex. 3. What do they become, if the altitude equals the diameter of the base?

Ex. 4. In a cylinder of revolution, what is the ratio of the lateral area to the area of the base? to the total area?

Proposition IV. Theorem

698. *The volume of a circular cylinder is equal to the product of its base by its altitude.*



Given the circular cylinder AJ, having its volume denoted by V, its base by B, and its altitude by H.

To prove $V = B \times H$.

Proof. Let a prism having a regular polygon for its base be inscribed in the cylinder, and denote the volume of the inscribed prism by V', and its base by B'.

The prism will have the same altitude, H, as the cylinder.

$\therefore \; V' = B' \times H.$ (Why?)

If the number of lateral faces of the inscribed prism be indefinitely increased,

V' will approach V as a limit. Art. 689, 1.

B' will approach B as a limit. (Why?)

And $B' \times H$ will approach $B \times H$ as a limit (Why?)

But $V' = B' \times H$ always. (Why?)

$\therefore \; V = B \times H.$ (Why?)

Q. E. D.

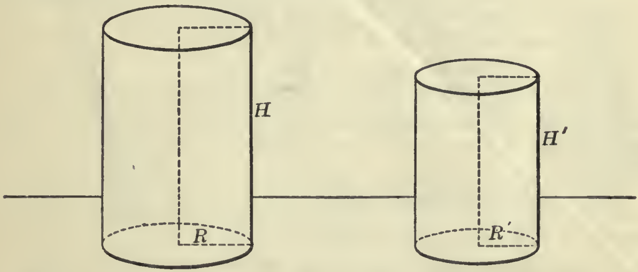
699. Formula for the volume of a circular cylinder. By use of Art. 450,

$$V = \pi R^2 H.$$

×

Proposition V. Theorem

700. *The lateral areas, or the total areas, of two similar cylinders of revolution are to each other as the squares of their radii, or as the squares of their altitudes; and their volumes are to each other as the cubes of their radii, or as the cubes of their altitudes.*



Given two similar cylinders of revolution having their lateral areas denoted by S and S', their total areas by T and T', their volumes by V and V', their radii by R and R', and their altitudes by H and H', respectively.

To prove $S : S' = T : T' = R^2 : R'^2 = H^2 : H'^2$;
and $V : V' = R^3 : R'^3 = H^3 : H'^3$.

Proof. $\dfrac{H}{H'} = \dfrac{R}{R'} = \dfrac{H+R}{H'+R'}$. Arts. 321, 312

$$\frac{S}{S'} = \frac{2\pi RH}{2\pi R'H'} = \frac{R \times H}{R' \times H'} = \frac{R}{R'} \times \frac{H}{H'} = \frac{R^2}{R'^2} = \frac{H^2}{H'^2}. \quad \text{(Why?)}$$

Also $$\frac{T}{T'} = \frac{2\pi R(H+R)}{2\pi R'(H'+R')} = \frac{R}{R'} \times \frac{H+R}{H'+R'} = \frac{R^2}{R'^2} = \frac{H^2}{H'^2}.$$ (Why?)

Also $$\frac{V}{V'} = \frac{\pi R^2 H}{\pi R'^2 H'} = \frac{R^2}{R'^2} \times \frac{H}{H'} = \frac{R^3}{R'^3} = \frac{H^3}{H'^3}. \quad \text{(Why?)}$$

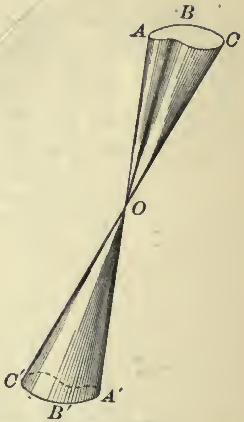
Q. E. D.

Ex. If a cylindrical cistern is 12 ft. deep, how much more cement is required to line it than to line a similar cistern 6 ft. deep? How much more water will the former cistern hold?

CONES

701. A **conical surface** is a surface generated by a straight line which moves so as to constantly touch a given fixed curve, and constantly pass through a given fixed point.

Thus every shadow cast by a near point of light is conical in form, that is, is bounded by a conical surface of light. Hence, the study of conical surfaces and solids is important from the fact that it concerns all cases of forces radiating from a near point.



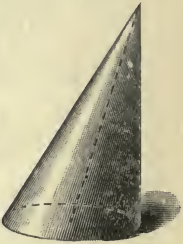
Conical surface

702. The **generatrix** of a conical surface is the moving straight line, as AA'; the **directrix** is the given fixed curve, as ABC; the **vertex** is the fixed point, as O; an **element** is the generatrix in any one of its positions, as BB'.

703. The **upper and lower nappes** of a conical surface are the portions above and below the vertex, respectively, as $O\text{–}ABC$ and $O\text{–}A'B'C'$.

Usually it is convenient to limit a conical surface to a single nappe.

704. A **cone** is a solid bounded by a conical surface and a plane cutting all the elements.

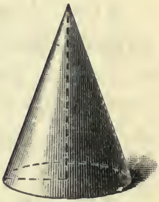

Oblique circular cone

705. The **base** of a cone is the face formed by the cutting plane; the **lateral surface** is the bounding conical surface; the **vertex** of the cone is the vertex of the conical surface; the **elements** of the cone are the elements of the conical surface; the **altitude** of a cone is the perpendicular distance from the vertex to the plane of the base.

706. A **circular cone** is a cone whose base is a circle. The **axis** of a circular cone is the line drawn from the vertex to the center of the base.

707. A **right circular cone** is a circular cone whose axis is perpendicular to the plane of the base.

An **oblique circular cone** is a circular cone whose axis is oblique to the base.



Cone of revolution

708. A **cone of revolution** is a cone generated by the revolution of a right triangle about one of its legs as an axis.

Hence a cone of revolution and a right circular cone are the same solid.

709. Properties of a cone of revolution inferred immediately.

1. *The altitude of a cone of revolution is the axis of the cone.*

2. *All the elements of a cone of revolution are equal.*

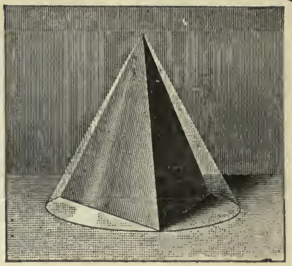
710. The **slant height of a cone of revolution** is any one of its elements.

711. Similar cones of revolution are cones generated by similar right triangles revolving about homologous sides.

712. A **plane tangent to a cone** is a plane which contains one element of the cone, but which does not cut the conical surface on being produced.

Ex. 1. A plane passing through a tangent to the base of a circular cone and the element drawn through the point of contact is tangent to the cone.

Ex. 2. If a plane is tangent to a circular cone, its intersection with the plane of the base is tangent to the cone.



713. A **pyramid inscribed in a cone** is a pyramid whose lateral edges are elements of the cone and whose base is a polygon inscribed in the base of the cone.

714. A **pyramid circumscribed about a cone** is a pyramid whose lateral faces are tangent to the cone and whose base is a polygon circumscribed about the base of the cone.

715. Properties of circular cones. By Art. 441 the area of a circle is the limit of the area of an inscribed, or of a circumscribed polygon, and the circumference is the limit of the perimeters of these polygons; hence

1. *The volume of a circular cone is the limit of the volume of an inscribed or circumscribed pyramid.*

2. *The lateral area of a circular cone is the limit of the lateral area of an inscribed or circumscribed pyramid.*

716. A **frustum of a cone** is the portion of the cone included between the base of the cone and a plane parallel to the base.

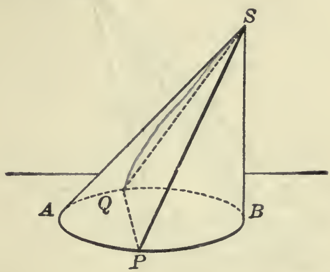
Frustum of a cone.

The **lower base of the frustum** is the base of the cone, and the **upper base of the frustum** is the section made by the plane parallel to the base of the cone.

What must be the **altitude and the lateral surface** of a frustum of a cone; also the **slant height** of the frustum of a cone of revolution?

Proposition VI. Theorem

717. *Every section of a cone made by a plane passing through its vertex is a triangle* and through an element



Given the cone S–$APBQ$ with a plane passing through the vertex S, and making the section SPQ.

To prove SPQ a triangle.

Proof. PQ, the intersection of the base and the cutting plane, is a straight line. (Why?)
SP is a straight line being an element

Draw the straight lines SP and SQ.

Then SP and SQ must be in the cutting plane; Art. 498.

And be elements of the conical surface. Art. 701.

$\therefore$ the straight lines SP and SQ are the intersections of the conical surface and the cutting plane.

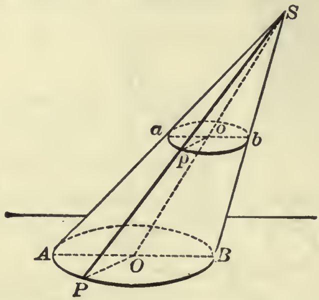
$\therefore$ the section SPQ is a triangle, Art. 81.
(*for it is bounded by three straight lines*).

Q. E. D.

Ex. What kind of triangle is a section of a right circular cone made by a plane through the vertex?

Proposition VII. Theorem

718. *Every section of a circular cone made by a plane parallel to the base is a circle.*



Given the circular cone SAB with apb a section made by a plane parallel to the base.

To prove apb a circle.

Proof. Denote the center of the base by O, and draw the axis, SO, piercing the plane of the section in o.

Through SO and any element, SP, of the conical surface, pass a plane cutting the plane of the base in the radius OP, and the plane of the section in op.

In like manner, pass a plane through SO and SB forming the intersections OB and ob.

$\therefore\ OP \parallel op$, and $OB \parallel ob$. (Why?)

$\therefore$ △ SPO and SBO are similar to △ Spo and Sbo, respectively. Art. 328.

$$\therefore \frac{op}{OP} = \left(\frac{So}{SO}\right) = \frac{ob}{OB}. \qquad \text{(Why?)}$$

But $OP = OB$. (Why?)

$\therefore\ op = ob$. (Why?)

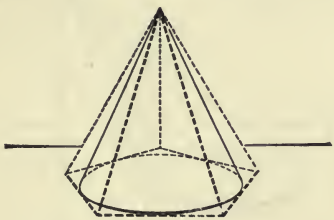
$\therefore$ opb is a circle. (Why?)

Q. E. D.

719. Cor. *The axis of a circular cone passes through the center of every section parallel to the base.*

PROPOSITION VIII. THEOREM

720. *The lateral area of a cone of revolution is equal to half the product of the slant height by the circumference of the base.*



Given a cone of revolution having its lateral area denoted by S, its slant height by L, and the circumference of its base by C.

To prove $S = \frac{1}{2}\, C \times L$.

Proof. Let a regular pyramid be circumscribed about the cone.

Denote the lateral area of the pyramid by S', and the perimeter of its base by P.

Then $S' = \frac{1}{2}\, P \times L$. Art. 642.

If the number of lateral faces of the circumscribed pyramid be indefinitely increased,

S' will approach S as a limit. Art. 715, 2.

P will approach C as a limit. Art. 441.

And $\frac{1}{2}\, P \times L$ will approach $\frac{1}{2}\, C \times L$ as a limit. Art. 253, 2.

But $S' = \frac{1}{2}\, P \times L$ always. (Why ?)

$\therefore\ S = \frac{1}{2}\, C \times L$. (Why ?)

Q. E. D.

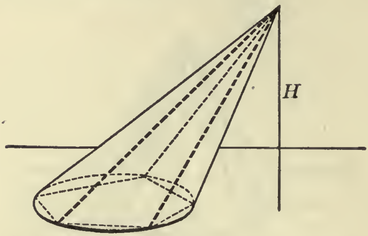
721. Formulas for lateral area and total area of a cone of revolution. Denoting the radius of the base by R,

$$S = \tfrac{1}{2}\,(2\,\pi R \times L) \quad \therefore\ S = \pi R L.$$

Also $$T = \pi R L + \pi R^2 \quad \therefore\ T = \pi R\,(L + R).$$

Proposition IX. Theorem

722. *The volume of a circular cone is equal to one-third of the product of its base by its altitude.*



Given a circular cone having its volume denoted by V, its base by B, and its altitude by H.

To prove $V = \frac{1}{3} B \times H$.

Proof. Let a pyramid with a regular polygon for its base be inscribed in the given cone.

Denote the volume of the inscribed pyramid by V', and its base by B'.

Hence $V' = \frac{1}{3} B' \times H$. Art. 651.

If the number of lateral faces of the inscribed pyramid be indefinitely increased,

V' will approach V as a limit. (Why?)

B' will approach B as a limit. (Why?)

And $\frac{1}{3} B' \times H$ will approach $\frac{1}{3} B \times H$ as a limit. (Why?)

But $V' = \frac{1}{3} B' \times H$ always. (Why?)

$\therefore V = \frac{1}{3} B \times H$. (Why?)

Q. E. D.

723. Formula for the volume of a circular cone.

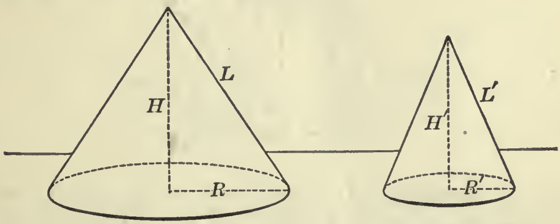
$$V = \frac{1}{3} \pi R^2 H.$$

Ex. 1. If, in a cone of revolution, $H=3$ and $R=4$, find S, T and V.

Ex. 2. If the altitude of a cone of revolution equals the radius of the base, what do the formulas for S, T and V become?

PROPOSITION X. THEOREM

724. *The lateral areas, or the total areas, of two similar cones of revolution are to each other as the squares of their radii, or as the squares of their altitudes, or as the squares of their slant heights; and their volumes are to each other as the cubes of these lines.*



Given two similar cones of revolution having their lateral areas denoted by S and S', their total areas by T and T', their volumes by V and V', their radii by R and R', their altitudes by H and H', and their slant heights by L and L', respectively.

To prove $S : S' = T : T' = R^2 : R'^2 = H^2 : H'^2 = L^2 : L'^2$; and $V : V' = R^3 : R'^3 = H^3 : H'^3 = L^3 : L'^3$.

Proof. $\dfrac{H}{H'} = \dfrac{R}{R'} = \dfrac{L}{L'} = \dfrac{L+R}{L'+R'}$. (Why?)

2. $\dfrac{S}{S'} = \dfrac{\pi RL}{\pi R'L'} = \dfrac{R}{R'} \times \dfrac{L}{L'} = \dfrac{R^2}{R'^2} = \dfrac{L^2}{L'^2} = \dfrac{H^2}{H'^2}$. (Why?)

3 $\dfrac{T}{T'} = \dfrac{\pi R\,(L+R)}{\pi R'\,(L'+R')} = \dfrac{R}{R'} \times \dfrac{L+R}{L'+R'} = \dfrac{R^2}{R'^2} = \dfrac{L^2}{L'^2} = \dfrac{H^2}{H'^2}$. (Why?)

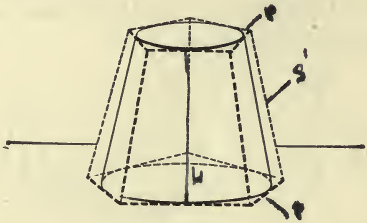
4 $\dfrac{V}{V'} = \dfrac{\frac{1}{3}\pi R^2 H}{\frac{1}{3}\pi R'^2 H'} = \dfrac{R^2}{R'^2} \times \dfrac{H}{H'} = \dfrac{R^3}{R'^3} = \dfrac{H^3}{H'^3} = \dfrac{L^3}{L'^3}$. (Why?)

Q. E. D.

725. DEF. An **equilateral cone** is a cone of revolution such that a section through the axis is an equilateral triangle.

Proposition XI. Theorem

726. *The lateral area of a frustum of a cone of revolution is equal to one-half the sum of the circumferences of its bases multiplied by its slant height.*



Given a frustum of a cone of revolution having its lateral area denoted by S, its slant height by L, the radii of its bases by R and r, and the circumferences of its bases by C and c.

To prove $S = \frac{1}{2}(C + c) \times L.$

Proof. Let the frustum of a regular pyramid be circumscribed about the given frustum. Denote the lateral area of the circumscribed frustum by S', the perimeter of the lower base by P, and the perimeter of the upper base by p.

The slant height of the circumscribed frustum is L.

Hence $S' = \frac{1}{2}(P + p) \times L.$ Art. 643.

Let the pupil complete the proof.

Q. E. D.

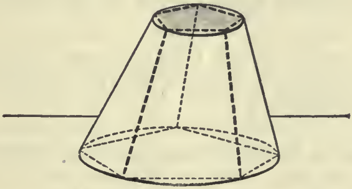
727. Formula for the lateral area of a frustum of a cone of revolution. $S = \frac{1}{2}(2\pi R + 2\pi r) L.$

$$\therefore S = \pi (R + r) L.$$

728. Cor. *The lateral area of a frustum of a cone of revolution is equal to the product of the circumference of its midsection by its slant height.*

Proposition XII. Theorem

729. *The volume of the frustum of a circular cone is equivalent to the volume of three cones, whose common altitude is the altitude of the frustum, and whose bases are the lower base, the upper base, and a mean proportional between the two bases.*



Given a frustum of a circular cone having its volume denoted by V, its altitude by H, the area of its lower base by B, and that of its upper base by b.

To prove $V = \frac{1}{3} H (B + b + \sqrt{B \times b})$.

Proof. Let the frustum of a pyramid with regular polygons for its bases be inscribed in the given frustum. Denote the volume of the inscribed frustum by V', and the areas of its bases by B' and b'.

$\therefore V' = \frac{1}{3} H (B' + b' + \sqrt{B' \times b'})$. (Why?)

If the number of lateral faces of the inscribed frustum be indefinitely increased, V' will approach V, B' and b' approach B and b respectively, and $B' \times b'$ approach $B \times b$, as limits. Art. 715.

Hence, also, $B' + b' + \sqrt{B' \times b'}$ will approach $B + b + \sqrt{B \times b}$ as a limit. Art. 253.

But $V' = \frac{1}{3} H (B' + b' + \sqrt{B' \times b'})$ always. (Why?)

$\therefore V = \frac{1}{3} H (B + b + \sqrt{B \times b})$. (Why?)

Q. E. D.

730. Formula for the volume of the frustum of a circular cone.

$$V = \tfrac{1}{3} H \left(\pi R^2 + \pi r^2 + \sqrt{\pi R^2 \times \pi r^2}\right).$$

$$\therefore V = \tfrac{1}{3} \pi H \left(R^2 + r^2 + Rr\right).$$

Ex. 1. The measurement of the volume of a frustum of a cone of revolution reduces to the measurement of the lengths of what straight lines?

Ex. 2. If a conical oil-can is 12 in. high, how much more tin is required to make it than to make a similar oil-can 6 in. high? How much more oil will it hold?

Ex. 3. The linear dimensions of a conical funnel are three times those of a similar funnel. How much more tin is required to make the first? How much more liquid will it hold?

Ex. 4. Make a similar comparison of cylindrical oil-tanks. Of conical canvas tents.

EXERCISES. GROUP 70

THEOREMS CONCERNING CYLINDERS AND CONES

Ex. 1. Any section of a cylinder of revolution through its axis is a rectangle.

Ex. 2. On a cylindrical surface only one straight line can be drawn through a given point.

[SUG. For if two straight lines could be drawn, etc.]

Ex. 3. The intersection of two planes tangent to a cone is a straight line through the vertex.

Ex. 4. If two planes are tangent to a cylinder, their line of intersection is parallel to an element of the cylinder.

[SUG. Pass a plane $\perp$ to the elements of the cylinder.]

Ex. 5. If tangent planes be passed through two diametrically opposite elements of a circular cone, these planes intersect in a straight line through the vertex and parallel to the plane of the base.

Ex. 6. In a cylinder of revolution the diameter of whose base equals the altitude, the volume equals one-third the product of the total surface by the radius of the base.

WRITE OUT

Ex. 7. A cylinder of revolution and a cone of revolution have the same base and the same altitude. Find the ratio of their lateral surfaces, and also of their volumes.

Ex. 8. If an equilateral triangle whose side is a be revolved about one of its sides as an axis, find the area generated in terms of a.

Ex. 9. If a rectangle whose sides are a and b be revolved first about the side a as an axis, and then about the side b, find the ratio of the lateral areas generated, and also of the volumes.

Ex. 10. The bases of a cylinder and of a cone of revolution are concentric. The two solids have the same altitude, and the diameter of the base of the cone is twice the diameter of the base of the cylinder. What kind of line is the intersection of their lateral surfaces, and how far is it from the base?

Ex. 11. Determine the same when the radius of the cone is three times the radius of the cylinder. Also when r times.

Ex. 12. Obtain a formula in terms of r for the volume of the frustum of an equilateral cone, in which the radius of the upper base is r and that of the lower base is $3r$.

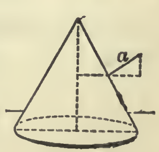
Ex. 13. A regular hexagon whose side is a revolves about a diagonal through the center as axis. Find, in terms of a, the surface and volume generated.

Ex. 14. Find the locus of a point at a given distance from a given straight line.

Ex. 15. Find the locus of a point whose distance from a given line is in a given ratio to its distance from a fixed plane perpendicular to the line.

Ex. 16. Find the locus of all straight lines which make a given angle with a given line at a given point.

Ex. 17. Find the locus of all straight lines which make a given angle with a given plane at a given point.



Ex. 18. Find the locus of all points at a given distance from the surface of a given cylinder of revolution.

Ex. 19. Find the locus of all points at a given distance from the surface of a given cone of revolution.

EXERCISES. GROUP 71

PROBLEMS CONCERNING THE CYLINDER AND CONE

Ex. 1. Through a given element of a circular cylinder, pass a plane tangent to the cylinder.

Ex. 2. Through a given element of a circular cone, pass a plane tangent to the cone.

Ex. 3. About a given circular cylinder circumscribe a prism, with a regular polygon for its base.

Ex. 4. Through a given point outside a circular cylinder, pass a plane tangent to the cylinder.

Ex. 5. Through a given point outside a given circular cone, pass a plane tangent to the cone.

[Sug. Through the vertex of the cone and the given point pass a line, and produce it to meet the plane of the base.]

Ex. 6. Into what segments must the altitude of a cone of revolution be divided by a plane parallel to the base, in order that the volume of the cone be bisected?

Ex. 7. Divide the lateral surface of a given cone of revolution into two equivalent parts by a plane parallel to the base.

Ex. 8. If the lateral surface of a cylinder of revolution be cut along one element and unrolled, what sort of a plane figure is formed?

Hence, out of cardboard construct a cylinder of revolution with given altitude and given circumference.

Ex. 9. If the lateral surface of a cone of revolution be cut along one element and unrolled, what sort of a plane figure is formed?

Hence, out of cardboard construct a cone of revolution of given slant height.

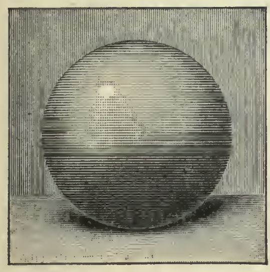
Ex. 10. Construct an equilateral cone out of pasteboard.

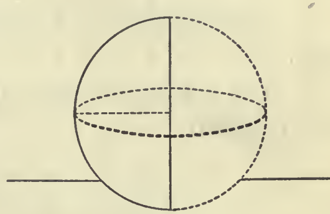
Ex. 11. Construct a frustum of a cone of revolution out of pasteboard.

Book IX

THE SPHERE

731. A **sphere** is a solid bounded by a surface all points of which are equally distant from a point within called the **center.**





732. A **sphere** may also be defined as a solid generated by the revolution of a semicircle about its diameter as an axis.

Some of the properties of a sphere may be obtained more readily from one of the two definitions given, and some from the other.

A sphere is named by naming the point at its center, or by naming three or more points on its surface.

733. A **radius of a sphere** is a line drawn from the center to any point on the surface.

A **diameter of a sphere** is a line drawn through the center and terminated at each end by the surface of the sphere.

734. A **line tangent to a sphere** is a line having but one point in common with the surface of the sphere, however far the line be produced.

735. **A plane tangent to a sphere** is a plane having but one point in common with the surface of the sphere, however far the plane be produced.

736. **Two spheres tangent to each other** are spheres whose surfaces have one point, and only one, in common.

737. **Properties of a sphere inferred immediately.**

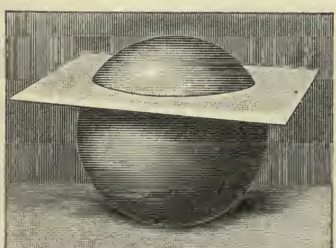
1. *All radii of a sphere, or of equal spheres, are equal.*

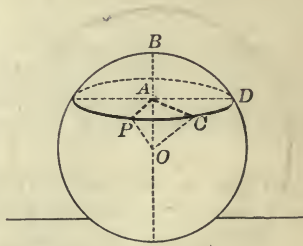
2. *All diameters of a sphere, or of equal spheres, are equal.*

3. *Two spheres are equal if their radii or their diameters are equal.*

PROPOSITION I. THEOREM

738. *A section of a sphere made by a plane is a circle.*





Given the sphere O, and PCD a section made by a plane cutting the sphere.

To prove that PCD is a circle.

Proof. From the center O, draw $OA \perp$ the plane of the section.

Let C be a fixed point on the perimeter of the section, and P any other point on this perimeter.

Draw AC, AP, OC, OP.

Then the $\triangle$ OAP and OAC are rt. $\triangle$. Art. 505.

$OP = OC$. (Why?)

$OA = OA$. (Why?)

$\triangle OAP = \triangle OAC$. (Why?)

$\therefore AP = AC$. (Why?)

But P is any point on the perimeter of the section PCD.

$\therefore$ every point on this perimeter is at the distance AC from A.

$\therefore$ PCD is a circle with center A. Art. 197.

Q. E. D.

739. Cor. 1. *Circles which are sections of a sphere made by planes equidistant from the center are equal; and conversely.*

740. Cor. 2. *Of two circles on a sphere, the one made by a plane more remote from the center is smaller; and conversely.*

741. Def. A **great circle of a sphere** is a circle whose plane passes through the center of the sphere.

742. Def. A **small circle of a sphere** is a circle whose plane does not pass through the center of the sphere.

743. Def. The **axis of a circle of a sphere** is the diameter of a sphere which is perpendicular to the plane of the circle. Thus, on figure p. 426, BB' is the axis of PCD.

744. Def. The **poles of a circle of a sphere** are the extremities of the axis of the circle. Thus, B and B', of figure p. 426, are poles of the circle PCD.

745. Properties of circles of a sphere inferred immediately.

1. *The axis of a circle of a sphere passes through the center of the circle; and conversely.*

2. *Parallel circles have the same axis and the same poles.*

3. *All great circles of a sphere are equal.*

The plane

4. of *Every great circle on a sphere bisects the sphere and its surface.*

5. *Two great circles on a sphere bisect each other.*

For the line of intersection of the two planes of the circles passes through the center, and hence is a diameter of each circle.

6. *Through two points (not the extremities of a diameter) on the surface of a sphere, one, and only one, great circle can be passed.*

For the plane of the great circle must also pass through the center of the sphere (Art. 741), and through three points not in a straight line only one plane can be passed (Art. 500).

7. *Through any three points on the surface of a sphere, not in the same plane with the center, one small circle, and only one, can be passed.*

746. Def. **The distance between two points on the surface of a sphere** is the length of the minor arc of a great circle joining the points.

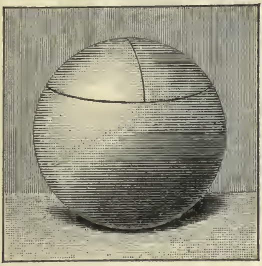
Ex. 1. If the radius of a sphere is 13 in., find the radius of a circle on the sphere made by a plane at a distance of 1 ft. from the center.

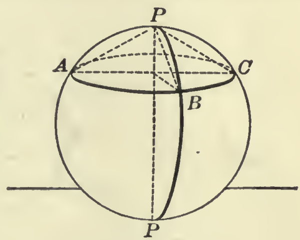
Ex. 2. What geographical circles on the earth's surface are great, and what small circles?

Ex. 3. What is the largest number of points in which two circles on the surface of a sphere can intersect? Why?

Proposition II. Theorem

747. *All points in the circumference of a circle of a sphere are equally distant from each pole of the circle.*





Given ABC a circle of a sphere, and P and P' its poles.

To prove the arcs PA, PB, PC equal, and arcs $P'A$, $P'B$, $P'C$ equal.

Proof. Draw the chords PA, PB, PC.

The chords PA, PB and PC are equal. Art. 518.

$\therefore$ arcs PA, PB and PC are equal. Art. 218.

In like manner, the arcs $P'A$, $P'B$ and $P'C$ may be proved equal.

Q. E. D.

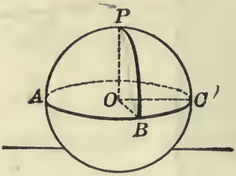
748. Def. The **polar distance of a small circle on a sphere** is the distance of any point on the circumference of the circle from the nearer pole.

The **polar distance of a great circle on a sphere** is the distance of any point on the circumference of the great circle from either pole.

749. Cor. *The polar distance of a great circle is the quadrant of a great circle.*

Proposition III. Theorem

750. *If a point on the surface of a sphere is at a quadrant's distance from two other points on the surface, it is the pole of the great circle through those points.*



Given PB and PC quadrants on the surface of the sphere O, and ABC a great circle through B and C.

To prove that P is the pole of ABC.

Proof. From the center O draw the radii OB, OC, OP.

The arcs PB and PC are quadrants. (Why?)

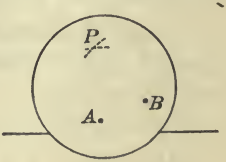
$\therefore$ ∠s POB and POC are rt. ∠s. (Why?)

$\therefore PO \perp$ plane ABC. (Why?)

$\therefore P$ is the pole of the great circle ABC. (Why?)

Q. E. D.

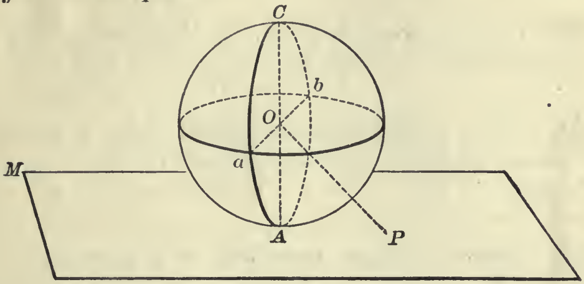
751. Cor. *Through two given points on the surface of a sphere to describe a great circle.*



Let A and B be the given points. From A and B as centers, with a quadrant as radius, describe arcs on the surface of the sphere intersecting at P. With P as a center and a quadrant as a radius, describe a great circle.

PROPOSITION IV. THEOREM

752. *A plane perpendicular to a radius at its extremity is tangent to the sphere.*



Given the sphere O, and the plane $MN \perp$ the radius OA of the sphere at its extremity A.

To prove MN tangent to the sphere.

Proof. Take P any point in plane MN except A. Draw OP. Then $OP > OA$. (Why?)

$\therefore$ the point P is outside the surface of the sphere.

But P is any point in the plane MN except A.

$\therefore$ plane MN is tangent to the sphere at the point A,

(for every point in the plane, except A, is outside the surface of the sphere).

Art. 735. **Q. E. D.**

753. COR. 1. *A plane, or a line, which is tangent to a sphere, is perpendicular to the radius drawn to the point of contact.* Also, *if a plane is tangent to a sphere, a perpendicular to the plane at its point of contact passes through the center of the sphere.*

754. COR. 2. *A straight line perpendicular to a radius of a sphere at its extremity is tangent to the sphere.*

755. COR. 3. *A straight line tangent to a circle of a sphere lies in the plane tangent to the sphere at the point of contact.*

756. Cor. 4. *A straight line drawn in a tangent plane, and through the point of contact is tangent to the sphere at that point.*

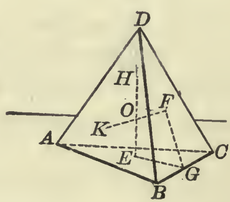
757. Cor. 5. *Two straight lines tangent to a sphere at a given point determine the tangent plane at that point.*

758. Def. **A sphere circumscribed about a polyhedron** is a sphere in whose surface lie all the vertices of the polyhedron.

759. Def. **A sphere inscribed in a polyhedron** is a sphere to which all the faces of the polyhedron are tangent.

Proposition V. Problem

760. *To circumscribe a sphere about a given tetrahedron.*



Given the tetrahedron $ABCD$.

To circumscribe a sphere about $ABCD$.

Construction and **Proof.** Construct E and F the centers of circles circumscribed about the ⊿ ABC and BCD, respectively. Art. 286.

Draw $EH \perp$ plane ABC and $FK \perp$ plane BCD. Art. 514.

Draw EG and FG to G the midpoint of BC.

Then EG and FG are $\perp$ BC. Art. 113.

$\therefore$ plane $EGF \perp BC$. Art. 509.

$\therefore$ plane $EGF \perp$ plane ABC. Art. 555.

$\therefore$ EH lies in the plane FGE. Art. 558.

In like manner FK lies in the plane FGE.

The lines EG and FG are not $\parallel$,

(*for they meet in the point* G).

$\therefore$ the lines EH and FK are not $\parallel$. Art. 122.

Hence EH must meet FK in some point O.

But EH is the locus of all points equidistant from A, B and C; and FK is the locus of all points equidistant from B, C and D. Art. 520.

$\therefore$ O, which is in both EH and FK, is equidistant from A, B, C and D. (Why?)

Hence a spherical surface constructed with O as a center and OA as a radius will pass through A, B, C and D, and form the sphere required. Q. E. F.

761. Cor. 1. *Four points not in the same plane determine a sphere.*

762. Cor. 2. *The four perpendiculars erected at the centers of the faces of a tetrahedron meet in a point.*

763. Cor. 3. *The six planes perpendicular to the edges of a tetrahedron at their midpoints intersect in a point.*

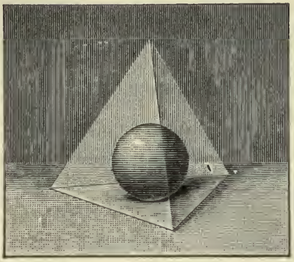
764. Def. An **angle formed by two curves** is the angle formed by a tangent to each curve at the point of intersection.

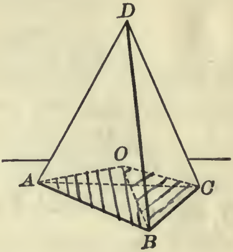
765. Def. A **spherical angle** is an angle formed by two intersecting arcs of great circles on a sphere, and hence by tangents to these arcs at the point of intersection.

Dup.

Proposition VI. Problem

766. *To inscribe a sphere in a given tetrahedron.*





Given the tetrahedron $ABCD$.

To inscribe a sphere in $ABCD$.

Construction and **Proof.** Bisect the dihedral angle D–AB–C by the plane OAB; similarly bisect the dihedral $\measuredangle$ whose edges are BC and AC by the planes OBC and OAC, respectively.

Denote the point in which the three bisecting planes intersect by O.

Every point in the plane OAB is equidistant from the faces DAB and CAB. Art. 562.

Similarly, every point in OBC is equidistant from the two faces intersecting in BC, and every point in OAC is equidistant from the two faces intersecting in AC.

$\therefore$ O is equidistant from all four faces of the tetrahedron. Ax. 1.

Hence, from O as a center, with the $\perp$ from O to any one face as a radius, describe a sphere.

This sphere will be tangent to the four faces of the tetrahedron and $\therefore$ inscribed in the tetrahedron. Art. 759.

Q. E. F.

767. Cor. *The planes bisecting the six dihedral angles of a tetrahedron meet in one point.*

Proposition VII. Problem

768. *To find the radius of a given material sphere.*

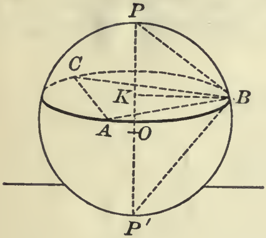


Fig. 1

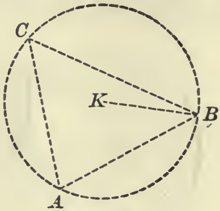


Fig. 2

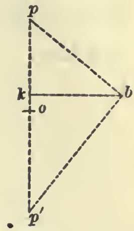


Fig. 3

Given the material sphere O.

To construct the radius of the sphere.

Construction. With any point P (Fig. 1) of the surface of the sphere as a pole, describe any convenient circumference on the surface.

On this circumference take any three points A, B and C.

Construct the $\triangle\ ABC$ (Fig. 2) having as sides the three chords AB, BC, AC, obtained from Fig. 1, by use of the compasses. Art. 283.

Circumscribe a circle about the $\triangle\ ABC$. Art. 286.

Let KB be the radius of this circle.

Construct (Fig. 3) the right $\triangle\ kpb$, having for hypotenuse the chord pb (Fig. 1) and the base kb. Art. 284.

Draw $bp' \perp bp$ and meeting pk produced at p'.

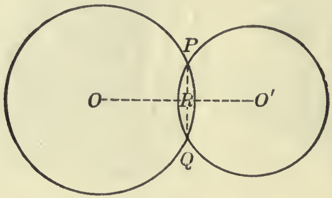
Bisect pp' at O.

Then op is the radius of the given sphere.

Proof. Let the pupil supply the proof. Q. E. F.

Proposition VIII. Theorem

769. *The intersection of two spherical surfaces is the circumference of a circle whose plane is perpendicular to the line joining the centers of the spheres, and whose center is in that line.*



Given two intersecting ⊙ O and O' which, by rotation about the line OO' as an axis, generate two intersecting spherical surfaces.

To prove that the intersection of the spherical surfaces is a ⊙, whose plane $\perp$ OO', and whose center lies in OO'.

Proof. Let the two circles intersect in the points P and Q, and draw the common chord PQ.

Then, as the two given ⊙ rotate about OO' as an axis, the point P will generate the line of intersection of the two spherical surfaces that are formed.

But PR is constantly $\perp$ OO'. Art. 241.

$\therefore$ PR generates a plane $\perp$ OO' Art. 510.

Also PR remains constant in length.

$\therefore$ P describes a circumference in that plane. Art. 197.

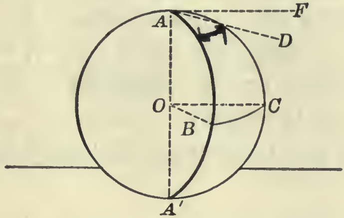
Hence the intersection of two spherical surfaces is a ⊙, whose plane $\perp$ the line of centers, and whose center is in the line of centers.

Q. E. D.

The above demonstration is an illustration of the use of the second definition of a sphere (Art. 732).

Proposition IX. Theorem

770. *A spherical angle is measured by the arc of a great circle described from the vertex of the angle as a pole, and included between its sides, produced, if necessary.*



Given $\angle BAC$ a spherical angle formed by the intersection of the arcs of the great circles BA and CA, and BC an arc of a great circle whose pole is A.

To prove $\angle BAC$ measured by arc BC.

Proof. Draw AD tangent to AB, and AF tangent to AC. Also draw the radii OB and OC.

Then $AD \perp AO$. Art. 230.

Also $OB \perp AO$ (*for AB is a quadrant*).

$\therefore OB \parallel AD$. (Why?)

Similarly $OC \parallel AF$.

$\therefore \angle BOC = \angle DAF$ Art. 538.

But $\angle BOC$ is measured by arc BC. Art. 257.

$\therefore \angle DAF$, that is, $\angle BAC$, is measured by arc BC. (Why?)

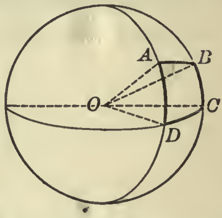
765

Q. E. D.

771. Cor. *A spherical angle is equal to the plane angle of the dihedral angle formed by the planes of its sides.*

SPHERICAL TRIANGLES AND POLYGONS

772. A **spherical polygon** is a portion of the surface of a sphere bounded by three or more arcs of great circles, as $ABCD$.



The **sides** of the spherical polygon are the bounding arcs; the **vertices** are the points in which the sides intersect; the **angles** are the spherical angles formed by the sides.

The sides of a spherical polygon are usually limited to arcs less than a semicircumference.

773. A **spherical triangle** is a spherical polygon of three sides.

Spherical triangles are classified in the same way as plane triangles; viz., as isosceles, equilateral, scalene, right, obtuse and acute.

774. Relation of spherical polygons to polyhedral angles. If radii be drawn from the center of a sphere to the vertices of a spherical polygon on its surface (as OA, OB, etc., in the above figure), a polyhedral angle is formed at O, which has an important relation to the spherical polygon $ABCD$

Each **face angle** *of the polyhedral angle equals (in number of degrees contained) the corresponding* **side** *of the spherical polygon;*

Each **dihedral angle** *of the polyhedral angle equals the corresponding* **angle** *of the spherical polygon.*

Hence, *corresponding to each property of a polyhedral angle, there exists a property of a spherical polygon, and conversely.*

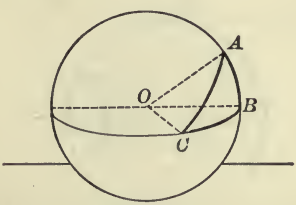
Hence, also, a *trihedral angle* and its parts correspond to a *spherical triangle* and its parts.

Of the common properties of a polyhedral angle and a spherical polygon, some are discovered more readily from the one figure and some from the other. In general, the spherical polygon is simpler to deal with than a polyhedral angle. For instance, if a trihedral angle were drawn with the plane angles of its dihedral angles, nine lines would be used, forming a complicated figure in solid space; whereas, the same magnitudes are represented in a spherical triangle by three lines in an approximately plane figure.

On the other hand, the spherical polygon, because of its lack of detailed parts, is often not so suggestive of properties as the polyhedral angle.

Proposition X. Theorem

775. *The sum of two sides of a spherical triangle is greater than the third side.*



Given the spherical triangle ABC, of which no side is larger than AB.

To prove $AC + BC > AB$.

Proof. From the center of the sphere, O, draw the radii OA, OB, OC.

Then, in the trihedral angle $O\text{-}ABC$,

$\angle AOC + \angle BOC > AOB$. Art. 582.

$\therefore AC + BC > AB$. Art. 774.

Q. E. D.

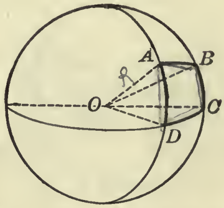
776. Cor. 1. *Any side of a spherical triangle is greater than the difference between the other two sides.*

777. Cor. 2. *The shortest path between two points on the surface of a sphere is the arc of the great circle joining those points.*

For any other path between the two points may be made the limit of a series of arcs of great circles connecting successive points on the path, and the sum of this series of arcs of great circles connecting the two points is greater than the single arc of a great circle connecting them.

Proposition XI. Theorem

778. *The sum of the sides of a spherical polygon is less than* 360°.



Given the spherical polygon $ABCD$.

To prove the sum of the sides of $ABCD < 360°$.

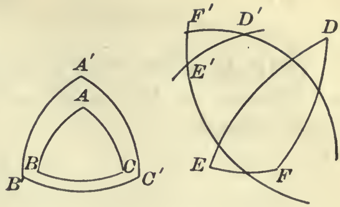
Proof. From O, the center of the sphere, draw the radii OA, OB, OC, OD.

Then $\angle AOB + \angle BOC + \angle COD + \angle DOA < 360°$. (Why ?)

$\therefore AB + BC + CD + DA < 360°$. Art. 774.

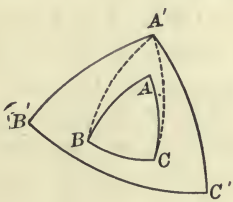
Q. E. D.

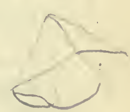
779. DEF. The **polar triangle** of a given triangle is the triangle formed by taking the vertices of the given triangle as poles, and describing arcs of great circles. (Hence, if each pole be regarded as a center, the radius used in describing each arc is a quadrant.) Thus $A'B'C'$ is the polar triangle of ABC; also $D'E'F'$ is the polar triangle of DEF.

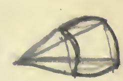


PROPOSITION XII. THEOREM

780. *If one spherical triangle is the polar of another, then the second triangle is the polar of the first.*







Given $A'B'C'$ the polar triangle of ABC.

To prove ABC the polar triangle of $A'B'C'$.

Proof. B is the pole of the arc $A'C'$. Art. 779.

$\therefore$ arc $A'B$ is a quadrant. (Why?)

Also C is the pole of the arc $A'B'$. (Why?)

$\therefore$ arc $A'C$ is a quadrant. (Why?)

$\therefore A'$ is at a quadrant's distance from both B and C.

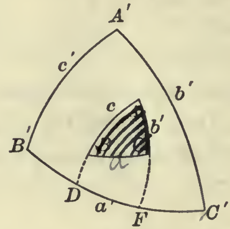
$\therefore A'$ is the pole of the arc BC. Art. 750.

In like manner it may be shown that B' is the pole of AC, and C' the pole of AB. Q. E. D.

Proposition XIII. Theorem

781. *In a spherical triangle and its polar, each angle of one triangle is the supplement of the side opposite in the other triangle.*

Imp.



Given the polar △ ABC and $A'B'C'$ with the sides of ABC denoted by a, b, c, and the sides of $A'B'C'$ denoted by a', b', c', respectively.

To prove $A + a' = 180°$, $B + b' = 180°$, $C + c' = 180°$,
$A' + a = 180°$, $B' + b = 180°$, $C' + c = 180°$.

Proof. Produce the sides AB and AC till they meet $B'C'$ in the points D and F, respectively.

Then B' is the pole of AF ∴ arc $B'F = 90°$. Art. 780.

Also C' is the pole of AD ∴ arc $C'D = 90°$. (Why?)

Adding, $B'F + C'D = 180°$. (Why?)

Or $B'F + FC' + DF = 180°$. Ax. 6.

Or $B'C' + DF = 180°$.

But $B'C' = a'$, and DF is the measure of the $\angle A$. Art. 770.

$\therefore A + a' = 180°$.

In like manner the other supplemental relations may be proved as specified.

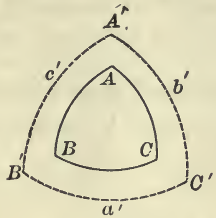
Q. E. D.

782. Def. **Supplemental triangles** are two spherical triangles each of which is the polar triangle of the other.

This new name for two polar triangles is due to the property proved in Art. 781.

Proposition XIV. Theorem

783. *The sum of the angles of a spherical triangle is greater than* 180°, *and less than* 540°.



Given the spherical triangle ABC.

To prove $A + B + C > 180°$ and $< 540°$.

Proof. Draw $A'B'C'$, the polar triangle of ABC, and denote its sides by a', b', c'.

Then
$$\left.\begin{aligned} A + a' &= 180° \\ B + b' &= 180° \\ C + c' &= 180° \end{aligned}\right\} \qquad \text{Art. 781.}$$

$$\therefore A + B + C + a' + b' + c' = 540°. \quad . \quad . \quad (1) \qquad \text{Ax. 2.}$$

But
$$\left\{\begin{aligned} a' + b' + c' &< 360° \\ a' + b' + c' &> 0° \end{aligned}\right. \qquad \text{Art. 778.}$$

Subtracting each of these in turn from (1),

$$A + B + C > 180° \text{ and } < 540°. \qquad \text{Ax. 11.}$$

Q. E. D.

784. Cor. *A spherical triangle may have one, two or three right angles; or it may have one, two or three obtuse angles.*

785. Def. A **birectangular spherical triangle** is a spherical triangle containing two right angles.

786. Def. A **trirectangular spherical triangle** is a spherical triangle containing three right angles.

787. Cor. *The surface of a sphere may be divided into eight trirectangular spherical triangles.* For let three planes ⊥ to each other be passed through the center of a sphere, etc.

788. Def. The **spherical excess** of a spherical triangle is the excess of the sum of its angles over 180°.

789. Def. **Symmetrical spherical triangles** are triangles which have their parts equal, but arranged in reverse order.

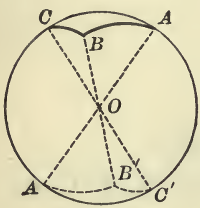


Fig. 1

Fig. 2

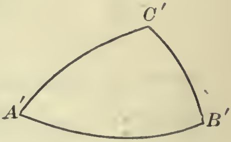
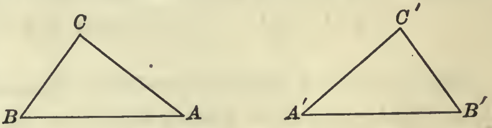


Fig. 3

Three planes passing through the center of a sphere form a pair of symmetrical spherical triangles on opposite sides of the sphere (see Art. 580), as $\triangle$ ABC and $A'B'C'$ of Fig. 1.

790. Equivalence of symmetrical spherical triangles. Two plane triangles which have their parts equal, but arranged in reverse order, may be made to coincide by lifting up one triangle, turning it over in space, and placing it upon the other triangle.

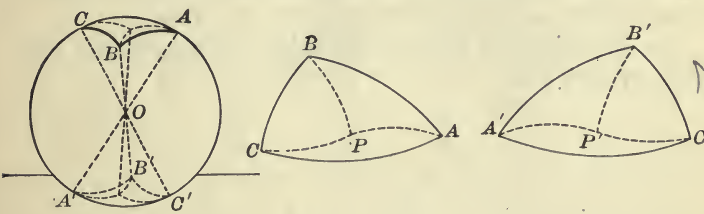


But two symmetrical spherical triangles cannot be made to coincide in this way, because of the curvature of a spherical surface. Hence the equivalence of two symmetrical spherical triangles must be demonstrated in some indirect way.

791. Property of symmetrical spherical triangles. *Two isosceles symmetrical spherical triangles are equal*, for they can be made to coincide.

Proposition XV. Theorem

792. *Two symmetrical spherical triangles are equivalent.*



Given the symmetrical spherical △ ABC and $A'B'C'$, formed by planes passing through O, the center of a sphere. (See Art. 789.)

To prove $\triangle ABC \approx \triangle A'B'C'$.

Proof. Let P be the pole of a small circle passing through the points A, B, C. Draw the diameter POP'.

Also draw PA, PB, PC, $P'A'$, $P'B'$, $P'C'$, all arcs of great ⊙.

Then $PA = PB = PC$. Art. 747.

Also $P'A' = PA$, $P'B' = PB$, $P'C' = PC$. Arts. 78, 215.

$\therefore P'A' = P'B' = P'C'$. Ax. 1.

Hence PAB and $P'A'B'$ are symmetrical isosceles △.

$\therefore \triangle PAB = \triangle P'A'B'$.
Similarly $\triangle PAC = \triangle P'A'C'$. Art. 791.
And $\triangle PBC = \triangle P'B'C'$.

Adding $\triangle PAB + \triangle PAC + \triangle PBC$
$\approx \triangle P'A'B' + \triangle P'A'C' + \triangle P'B'C'$. Ax. 2.

Or $\triangle ABC \approx \triangle A'B'C'$. Ax. 6.

In case the poles P and P' fall outside the △ ABC and $A'B'C'$, let the pupil supply the demonstration. Q. E. D.

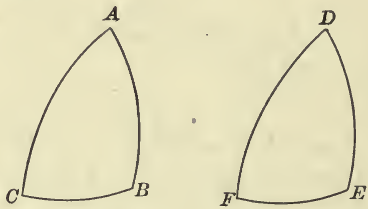
PROPOSITION XVI. THEOREM

793. *On the same sphere, or on equal spheres, two triangles are equal,*

I. *If two sides and the included angle of one are equal to two sides and the included angle of the other; or*

II. *If two angles and the included side of one are equal to two angles and the included side of the other,*

the corresponding equal parts being arranged in the same order in each case.



I. **Given** the spherical $\triangle$ ABC and DEF, in which $AC=DF$, $CB=FE$, and $\angle C=\angle F$.

To prove $\triangle ABC=\triangle DEF$.

Proof. Let the pupil supply the proof (see Book I, Prop. VI).

II. **Given** the spherical $\triangle$ ABC and DEF, in which $\angle C=\angle F$, $\angle B=\angle E$, and $CB=FE$.

To prove $\triangle ABC=\triangle DEF$.

Proof. Let the pupil supply the proof (see Book I, Prop. VII).

Ex. 1. If the line of centers of two spheres is 10 in., and the radii are 12 in. and 3 in., how are the spheres situated with reference to each other?

Ex. 2. The tank on a motor car is a cylinder 35 inches long and 15 inches in diameter. How many gallons of gasolene will it hold?

Ex. 3. In an equilateral cone, find the ratio of the lateral area to the area of the base.

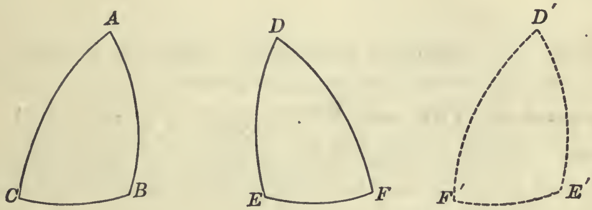
PROPOSITION XVII. THEOREM

794. *On the same sphere, or on equal spheres, two triangles are symmetrical and equivalent,*

I. *If two sides and the included angle of one are equal to two sides and the included angle of the other; or*

II. *If two angles and the included side of one are equal to two angles and the included side of the other,*

the corresponding equal parts being arranged in reverse order.



I. **Given** the spherical $\triangle$ ABC and DEF, in which $AB = DE$, $AC = DF$, and $\angle A = \angle D$, the corresponding parts being arranged in reverse order.

To prove $\triangle\ ABC$ symmetrical with $\triangle\ DEF$.

Proof. Construct the $\triangle D'E'F'$ symmetrical with $\triangle DEF$.

Then $\triangle ABC$ may be made to coincide with $\triangle\ D'E'F'$, Art. 793.

(*having two sides and the included $\angle$ equal and arranged in the same order*).

But $\triangle\ D'E'F'$ is symmetrical with the $\triangle\ DEF$.

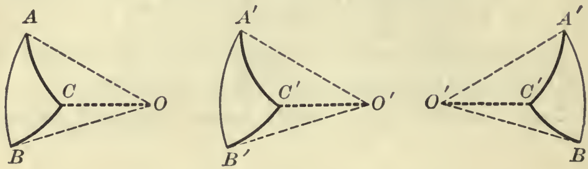
$\therefore$ $\triangle\ ABC$, which coincides with $\triangle\ D'E'F'$, is symmetrical with $\triangle\ DEF$.

II. The second part of the theorem is proved in the same way.

Q. E. D.

Proposition XVIII. Theorem

795. *If two triangles on the same sphere, or equal spheres, are mutually equilateral, they are also mutually equiangular, and therefore equal or symmetrical.*



Given two mutually equilateral spherical $\triangle s$ ABC and $A'B'C'$ on the same or on equal spheres.

To prove $\triangle s$ ABC and $A'B'C'$ equal or symmetrical.

Proof. From O and O', the centers of the spheres to which the given triangles belong, draw the radii OA, OB, OC, $O'A'$, $O'B'$, $O'C'$.

Then the face $\angle s$ at O = corresponding face $\angle s$ at O'. Art. 774.

Hence dihedral $\angle s$ at O = corresponding dihedral $\angle s$ at O'. Art. 584.

$\therefore$ $\angle s$ of spherical $\triangle$ ABC = homologous $\angle s$ of spherical $\triangle$ $A'B'C'$. Art. 774.

$\therefore$ the $\triangle s$ ABC and $A'B'C'$ are equal or symmetrical, according as their homologous parts are arranged in the same or in reverse order. Art. 789.

796. Note. The conditions in Props. XVI and XVIII which make two spherical triangles equal are the same as those which make two plane triangles equal. Hence many other propositions occur in spherical geometry which are identical with corresponding propositions in plane geometry. Thus, many of the construction problems of spherical geometry are solved in the same way as the corresponding construction problems in plane geometry; as, to bisect a given angle, etc.

Proposition XIX. Theorem

797. *If two triangles on the same sphere are mutually equiangular, they are also mutually equilateral, and therefore equal or symmetrical.*

Given the mutually equiangular spherical ⚠ Q and Q' on the same sphere or on equal spheres.

To prove that Q and Q' are mutually equilateral, and therefore equal or symmetrical.

Proof. Construct P and P' the polar ⚠ of Q and Q', respectively.

Then ⚠ P and P' are mutually equilateral. Art. 781.

∴ ⚠ P and P' are mutually equiangular. Art. 795.

But Q is the polar △ of P, and Q' of P'. Art. 780.

∴ ⚠ Q and Q' are mutually equilateral. Art. 781.

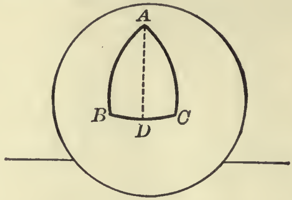
Hence Q and Q' are equal or symmetrical, according as their homologous parts are arranged in the same or in reverse order. Art. 789.

Q. E. D.

798. Cor. *If two mutually equiangular triangles are on unequal spheres, their corresponding sides have the same ratio as the radii of their respective spheres.*

Proposition XX. Theorem

799. *In an isosceles spherical triangle the angles opposite the equal sides are equal.*



Given the spherical $\triangle$ ABC in which $AB = AC$.

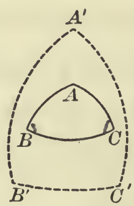
To prove $\angle B = \angle C$.

Proof. Draw an arc from the vertex A to D, the midpoint of the base.

Let the pupil supply the remainder of the proof.

Proposition XXI. Theorem (Conv. of Prop. XX)

800. *If two angles of a spherical triangle are equal, the sides opposite these angles are equal, and the triangle is isosceles.*



Given the spherical $\triangle$ ABC in which $\angle B = \angle C$.

To prove $AB = AC$.

Proof. Construct $\triangle$ $A'B'C'$ the polar $\triangle$ of ABC.

Then $A'C' = A'B'$. Art. 781.

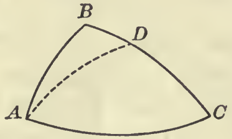
$\therefore \angle C' = \angle B'$. Art. 799.

$\therefore AB = AC$. Art. 781.

Q. E. D.

Proposition XXII. Theorem

801. *In any spherical triangle, if two angles are unequal, the sides opposite these angles are unequal, and the greater side is opposite the greater angle, and* Conversely.



Given the spherical $\triangle ABC$ in which $\angle BAC$ is greater than $\angle C$.

To prove $BC > BA$.

Proof. Draw the arc AD making $\angle DAC$ equal to $\angle C$.

Then $DA = DC$. Art. 800.

To each of these equals add the arc BD.

$\therefore BD + DA = BD + DC$, or BC. (Why?)

But in $\triangle BDA$, $BD + DA > BA$. (Why?)

$\therefore BC > BA$. Ax. 8.

Let the pupil prove the converse by the indirect method (see Art. 106).

Q. E. D.

Ex. 1. Bisect a given spherical angle.

Ex. 2. Bisect a given arc of a great circle on a sphere.

Ex. 3. At a given point in an arc on a sphere, construct an angle equal to a given spherical angle on the same sphere.

Ex. 4. Find the locus of the centers of the circles of a sphere formed by planes perpendicular to a given diameter of the given sphere.

SPHERICAL AREAS

802. Units of spherical surface. A spherical surface may be measured in terms of, either

1. The *customary units of area*, as a square inch, a square foot, etc., or

2. *Spherical degrees*, or *spherids*.

803. A **spherical degree, or spherid,** is one-ninetieth part of one of the eight trirectangular triangles into which the surface of a sphere may be divided (Art. 787), or $\frac{1}{720}$ part of the surface of the entire sphere.

A solid degree is one-ninetieth part of a trirectangular angle (see Art. 774).

804. A **lune** is a portion of the surface of a sphere bounded by two semicircumferences of great circles, as $PBP'C$ of Fig. 1.

The **angle of a lune** is the angle formed by the semicircumferences which bound it, as the angle BPC.

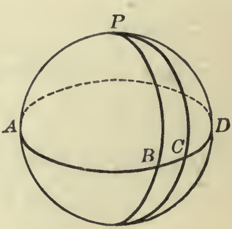


Fig. 1.

805. A **zone** is the portion of the surface of the sphere bounded by two parallel planes.

A zone may also be defined as the surface generated by an arc of a revolving semicircumference. Thus, if QFQ' (Fig. 2) generates a sphere by rotating about QQ', its diameter, any arc of QFQ', as EF, generates a zone.

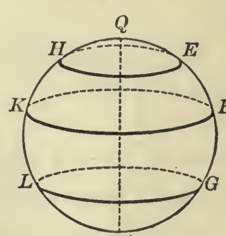


Fig. 2.

806. A **zone of one base** is a zone one of whose bounding planes is tangent to the sphere, as the zone generated by the arc QE of Fig. 2.

807. The **altitude** of a zone is the perpendicular distance between the bounding planes of the zone.

The **bases** of a zone are the circumferences of the circles of the sphere formed by the bounding planes of the zone.

PROPOSITION XXIII. THEOREM

808. *The area generated by a straight line revolving about an axis in its plane is equal to the projection of the line upon the axis, multiplied by the circumference of a circle whose radius is the perpendicular erected at the midpoint of the line and terminated by the axis.*

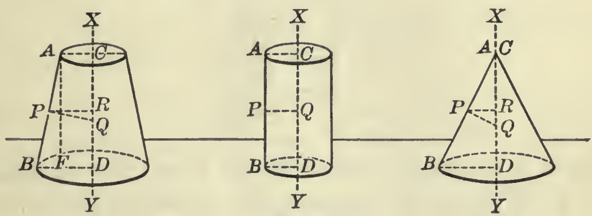


Fig. 1 Fig. 2 Fig. 3

Given AB and XY in the same plane, CD the projection of AB on XY, PQ the $\perp$ bisector of AB; and a surface generated by the revolution of AB about XY, denoted as "area AB."

To prove area $AB = CD \times 2\,\pi PQ$.

Proof. 1. In general, the surface generated by AB is the surface of a frustum of a cone (Fig. 1).

$\therefore$ area $AB = AB \times 2\,\pi PR$. Art. 728.

Draw $AF \perp BD$, then

$\triangle$ ABF and PQR are similar. Art. 328.

$\therefore AB : AF = PQ : PR$. (Why?)

$\therefore AB \times PR = AF \times PQ$, or $CD \times PQ$. (Why?)

Substituting, area $AB = CD \times 2\,\pi PQ$. Ax. 8.

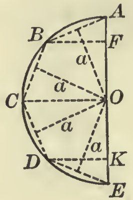
2. If $AB \parallel XY$ (Fig. 2), the surface generated by AB is the lateral surface of a cylinder.

$\therefore$ area $AB = CD \times 2\,\pi PQ$. Art. 697.

3. If the point A lies in the axis XY (Fig. 3), let the pupil show that the same result is obtained. Q. E. D.

Proposition XXIV. Theorem

809. *The area of the surface of a sphere is equal to the product of the diameter of the sphere by the circumference of a great circle.*



Given a sphere generated by the revolution of the semicircle ACE about the diameter AE, with the surface of the sphere denoted by S, and its radius by R.

To prove $S = AE \times 2\,\pi R$.

Proof. Inscribe in the given semicircle the half of a regular polygon of an even number of sides, as $ABCDE$.

Draw the apothem to each side of the semipolygon, and denote it by a.

From the vertices B, C, D draw $\perp$ to AE.

Then
$$\left.\begin{aligned} \text{area } AB &= AF \times 2\,\pi a. \\ \text{area } BC &= FO \times 2\,\pi a. \\ \text{area } CD &= OK \times 2\,\pi a. \\ \text{area } DE &= KE \times 2\,\pi a. \end{aligned}\right\} \qquad \text{Art. 808.}$$

Adding, area $ABCDE = AE \times 2\,\pi a$.

If, now, the number of sides of the polygon be indefinitely increased,

area $ABCDE$ approaches S as a limit. Art. 441.

And a approaches R as a limit. (Why?)

$\therefore AE \times 2\,\pi a$ approaches $AE \times 2\,\pi R$ as a limit. (Why?)

But area $ABCDE = AE \times 2\,\pi a$ always.

$\therefore S = AE \times 2\,\pi R$. (Why?)

Q. E. D.

810. Formulas for area of surface of a sphere.

Substituting for AE its equal $2R$, $S=4\pi R^2$.

Also denoting the diameter of the sphere by D, $R=\frac{1}{2}D$.

$$\therefore S=4\pi\left(\frac{D}{2}\right)^2, \text{ or } S=\pi D^2.$$

811. Cor. 1. *The surface of a sphere is equivalent to four times the area of a great circle of the sphere.*

812. Cor. 2. *The areas of the surfaces of two spheres are to each other as the squares of their radii, or of their diameters.*

For, if S and S' denote the surfaces, R and R' the radii, and D and D' the diameters of two spheres,

$$\frac{S}{S'}=\frac{4\pi R^2}{4\pi R'^2}=\frac{R^2}{R'^2}; \text{ also } \frac{S}{S'}=\frac{\pi D^2}{\pi D'^2}=\frac{D^2}{D'^2}.$$

813. Property of the sphere. The following property of the sphere is used in the proof of Art. 809: *If, in the generating arc of any zone, a broken line be inscribed, whose vertices divide the arc into equal parts, then, as the number of these parts is increased indefinitely, the area generated by the broken line approaches the area of the zone as a limit.* Hence

Cor. 3. *The area of a zone is equal to the circumference of a great circle multiplied by the altitude of the zone.*

Thus the area generated by the arc $BC = FO \times 2\pi R$.

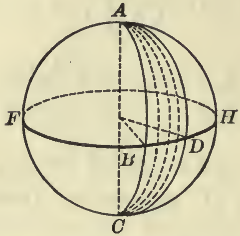
814. Cor. 4. *On the same sphere, or on equal spheres, the areas of two zones are to each other as the altitudes of the zones.*

Ex. 1. Find the area of a sphere whose diameter is 10 in.

Ex. 2. Find the area of a zone of altitude 3 in., on a sphere whose radius is 10 in.

Proposition XXV. Theorem

815. *The area of a lune is to the area of the surface of the sphere as the angle of the lune is to four right angles.*



Given a sphere having its area denoted by S, and on the sphere the lune $ABCD$ of $\angle A$ with its area denoted by L.

To prove $L : S = A^\circ : 360^\circ$.

Proof. Draw FBH, the great ⊙ whose pole is A, intersecting the bounding arcs of the lune in B and D.

Case I. *When the arc BD and the circumference FBH are commensurable.*

Find a common measure of BD and FBH, and let it be contained in the arc BD m times, and in the circumference FBH n times.

Then arc BD : circumference $FBH = m : n$.

Through the diameter AC, and the points of division of the circumference FBH pass planes of great ⊙.

The arcs of these great ⊙ will divide the surface of the sphere in n small equal lunes, m of them being contained in the lune $ABCD$.

$$\therefore L : S = m : n.$$

$\therefore L : S =$ arc BD : circumference FBH. (Why?)

Or $L : S = A^\circ : 360^\circ$. Art. 257.

Case II. *When the arc BD and the circumference FBH are incommensurable.*

Let the pupil supply the proof. Q. E. D.

816. Formula for the area of a lune in spherical degrees, or spherids. The surface of a sphere contains 720 spherids (Art. 803). Hence, by Art. 815,

$$\therefore \frac{L}{S} \text{ or } \frac{L \text{ spherids}}{720 \text{ spherids}} = \frac{A}{360}, \quad \therefore \frac{L}{2} = A, \boxed{\text{or } L = 2\,A \text{ spherids;}}$$

that is, *the area of a lune in spherical degrees is equal to twice the number of angular degrees in the angle of the lune.*

817. Formula for area of a lune in square units of

$$S = 4\,\pi R^2 \text{ (Art. 810)} \quad \therefore \frac{L}{4\,\pi R^2} = \frac{A}{360}, \text{ or } \boxed{L = \frac{\pi R^2 A}{90}.}$$

818. COR. 1. *On the same sphere, or on equal spheres, two lunes are to each other as their angles.*

819. COR. 2. *Two lunes with equal angles, but, on unequal spheres. are to each other as the squares of the radii of their spheres.*

$$\text{For } L : L' = \frac{\pi R^2 A}{90} : \frac{\pi R'^2 A}{90}, \text{ or } \boxed{L : L' = R^2 : R'^2}$$

Ex. 1. Find the area in spherical degrees of a lune of 27°.

Ex. 2. Find the number of square inches in the area of a lune of 27°, on a sphere whose radius is 10 in.

A solid symmetrical with respect to a plane is a solid in which a line drawn from any point in its surface ⊥ the given plane and produced its own length ends in a point on the surface; hence

Ex. 3. How many planes of symmetry has a circular cylinder? A cylinder of revolution?

Ex. 4. Has either of these solids a center of symmetry?

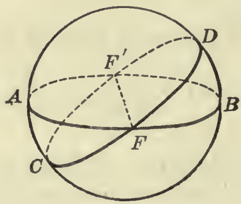
Ex. 5. Answer the same questions for a circular cone.

Ex. 6. For a cone of revolution. For a sphere.

Ex. 7. For a regular square pyramid. For a regular pentagonal pyramid.

Proposition XXVI. Theorem

820. *If two great circles intersect on a hemisphere, the sum of two vertical triangles thus formed is equivalent to a lune whose angle is that angle in the triangles which is formed by the intersection of the two great circles.*



Given the hemisphere $ADBF$, and on it the great circles AFB and DFC, intersecting at F.

To prove $\triangle AFC + \triangle BFD \approx$ lune whose $\angle$ is BFD.

Proof. Complete the sphere and produce the given arcs of the great circles to intersect at F' on the other hemisphere.

Then, in the $\triangle$s AFC and $BF'D$,

$$\text{arc } AF = \text{arc } BF',$$

(*each being the supplement of the arc BF*).

In like manner arc $CF =$ arc DF'.

And arc $AC =$ arc DB.

$\therefore \triangle AFC \approx \triangle BF'D.$ Art. 795.

Add the $\triangle BFD$ to each of these equals;

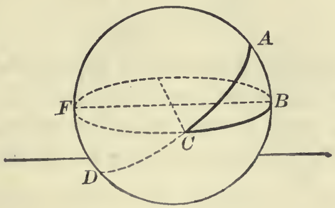
$\therefore \triangle AFC + \triangle BFD \approx \triangle BF'D + \triangle BFD.$ Ax. 3.

Or $\therefore \triangle AFC + \triangle BFD \approx$ lune $FBF'D$, Ax. 6

Q. E. D.

Proposition XXVII. Theorem

821. *The number of spherical degrees, or spherids, in the area of a spherical triangle is equal to the number of angular degrees in the spherical excess of the triangle.*



Given the spherical $\triangle$ ABC whose $\angle$s are denoted by A, B, C, and whose spherical excess is denoted by E.

To prove area of $\triangle$ $ABC = E$ spherids.

Proof. Produce the sides AC and BC to meet AB produced in the points D and F, respectively.

$\left.\begin{array}{l}\triangle ABC + \triangle CDB = \text{lune } ABDC = 2\,A \text{ spherids.}\\ \triangle ABC + \triangle ACF = \text{lune } BCFA = 2\,B \text{ spherids.}\end{array}\right\}$ Art. 816.

$\triangle ABC + \triangle CFD = \text{lune of } \angle BCA = 2\,C$ spherids. Art. 820.

Adding, and observing that $\triangle ABC + \triangle CDB + \triangle ACF + \triangle CFD =$ hemisphere $ABDFC$,

$2 \triangle ABC +$ hemisphere $= 2\,(A + B + C)$ spherids. Or,
$2 \triangle ABC + 360$ spherids $= 2\,(A + B + C)$ spherids. Ax. 8.

$\therefore \triangle ABC + 180$ spherids $= (A + B + C)$ spherids. Ax. 5.

$\therefore \triangle ABC = (A + B + C - 180)$ spherids. Ax. 3.

Or area $\triangle ABC = E$ spherids. Art. 788.

Q. E. D.

822. Formula for area of a spherical triangle in square units of area.

Comparing the area of a spherical △ with the area of the entire sphere,

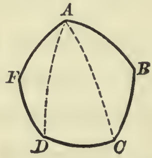
$$\text{area } \triangle : 4\,\pi R^2 = E \text{ spherids} : 720 \text{ spherids}.$$

$$\therefore \text{ area } \triangle = \frac{4\,\pi R^2 \times E}{720}, \text{ or area } \triangle = \frac{\pi R^2 E}{180}.$$

823. The **spherical excess of a spherical polygon** is the sum of the angles of the polygon diminished by $(n-2)\,180°$; that is, it is the sum of the spherical excesses of the triangles into which the polygon may be divided.

Proposition XXVIII. Theorem

824. *The area of a spherical polygon, in spherical degrees or spherids, is equal to the spherical excess of the polygon.*



Given a spherical polygon $ABCDF$ of n sides, with its spherical excess denoted by E.

To prove area of $ABCDF = E$ spherical degrees.

Proof. Draw diagonals from A, any vertex of the polygon, and thus divide the polygon into $n-2$ spherical △.

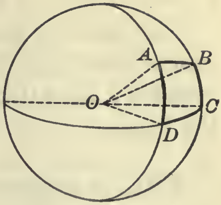
The area of each △ = (sum of its ∠ − 180) spherids. Art. 821.

∴ sum of the areas of the △ = [sum of ∠ of the △ − $(n-2)$ 180] spherids. Ax. 2.

∴ area of polygon $= E$ spherids, Art. 823.

(for the sum of ∠ of the △ − $(n-2)\,180° = E°$).

SPHERICAL VOLUMES

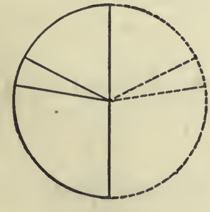


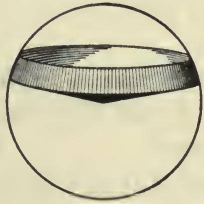
825. A **spherical pyramid** is a portion of a sphere bounded by a spherical polygon and the planes of the great circles forming the sides of the polygon. The **base** of a spherical pyramid is the spherical polygon bounding it, and the **vertex** of the spherical pyramid is the center of the sphere.

Thus, in the spherical pyramid $O\text{-}ABCD$, the base is $ABCD$ and the vertex is O.

826. A **spherical wedge** (or **ungula**) is the portion of a sphere bounded by a lune and the planes of the sides of the lune.

827. A **spherical sector** is the portion of a sphere generated by a sector of that semicircle whose rotation generates the given sphere.





828. The **base** of a spherical sector is the zone generated by the revolution of the arc of the plane sector which generates the spherical sector.

Let the pupil draw a spherical sector in which the base is a zone of one base.

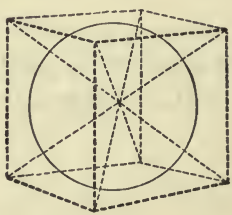
829. A **spherical segment** is a portion of a sphere included between two parallel planes.

The **bases** of a spherical segment are the sections of the sphere made by the parallel planes which bound the given segment; the **altitude** is the perpendicular distance between the bases.

830. A **spherical segment of one base** is a spherical segment one of whose bounding planes is tangent to the sphere.

Proposition XXIX. Theorem

831. *The volume of a sphere is equal to one-third the product of the area of its surface by its radius.*



Given a sphere having its volume denoted by V, surface by S, and radius by R.

To prove $V = \frac{1}{3} S \times R$.

Proof. Let any polyhedron be circumscribed about the sphere.

Pass a plane through each edge of the polyhedron and the center of the sphere.

These planes will divide the polyhedron into as many pyramids as the polyhedron has faces, each pyramid having a face of the polyhedron for its base, the center of the

sphere for its vertex, and the radius of the sphere for its altitude.

$\therefore$ volume of each pyramid $= \frac{1}{3}$ base $\times R$. (Why?)

$\therefore$ volume of polyhedron $= \frac{1}{3}$ (surface of polyhedron) $\times R$.

If the number of faces of the polyhedron be increased indefinitely, the volume of the polyhedron approaches the volume of the sphere as a limit, and the surface of the polyhedron approaches the surface of the sphere as a limit.

Hence the volume of the polyhedron and $\frac{1}{3}$ (surface of the polyhedron) $\times R$, are two variables always equal.

Hence their limits are equal,

Or $$V = \frac{1}{3} S \times R.$$ (Why?)

Q. E. D.

832. Formulas for volume of a sphere. Substituting $S = 4\pi R^2$, or $S = \pi D^2$ in the result of Art. 831,

$$V = \frac{4\pi R^3}{3}; \text{ also } V = \frac{\pi D^3}{6}.$$

833. COR. 1. *The volumes of two spheres are to each other as the cubes of their radii, or as the cubes of their diameters.*

For $$\frac{V}{V'} = \frac{\frac{4}{3}\pi R^3}{\frac{4}{3}\pi R'^3} = \frac{R^3}{R'^3}; \text{ also } \frac{V}{V'} = \frac{\frac{1}{6}\pi D^3}{\frac{1}{6}\pi D'^3} = \frac{D^3}{D'^3}.$$

834. COR. 2. *The volume of a spherical pyramid is equal to one-third the product of its base by the radius of the sphere.*

835. COR. 3. *The volume of spherical sector is equal to one-third the product of its base (the bounding zone) by the radius of the sphere.*

836. Formula for the volume of a spherical sector. Denoting the altitude of the sector by H and the volume by V,

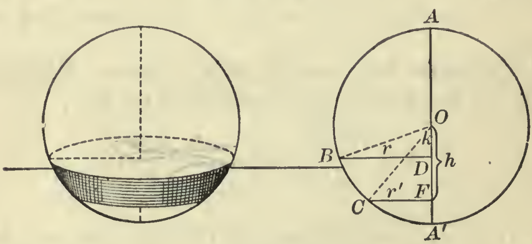
$$V=\tfrac{1}{3}\text{ (area of zone)}\times R,$$

$$=\tfrac{1}{3}(2\pi RH)R. \qquad \text{Art. 813.}$$

$$\therefore V=\tfrac{2}{3}\pi R^2H.$$

Proposition XXX. Theorem

837. *The volume of a spherical segment is equal to one-half the product of its altitude by the sum of the areas of its bases, plus the volume of a sphere whose diameter is the altitude of the segment.*



Given the semicircle $ABCA'$ which generates a sphere by its rotation about the diameter AA'; BD and CF semichords $\perp AA'$, and denoted by r and r'; and DF denoted by H.

To prove volume of spherical segment generated by $BCFD$, or $V=\frac{1}{2}(\pi r^2+\pi r'^2)H+\frac{1}{6}\pi H^3$.

Proof. Draw the radii OB and OC.

Denote OF by h, and OD by k.

Then $V=\text{vol. } OBC+\text{vol. } OCF-\text{vol. } OBD.$

$$\therefore V=\tfrac{2}{3}\pi R^2H+\tfrac{1}{3}\pi r'^2h-\tfrac{1}{3}\pi r^2k. \qquad \text{Arts. 836, 723.}$$

But $H=h-k$, $h^2=R^2-r'^2$, and $k^2=R^2-r^2$, (Why?)

$\therefore V=\frac{1}{3}\pi [2R^2(h-k)+(R^2-h^2)h-(R^2-k^2)k]$. Ax. 8.
$=\frac{1}{3}\pi [2R^2(h-k)+R^2(h-k)-(h^3-k^3)]$.
$=\frac{1}{3}\pi H[3R^2-(h^2+hk+k^2)]$.

But $h^2-2hk+k^2=H^2$. Ax. 4.

Subtract each member from $3h^2+3k^2$ and divide by 2. Axs. 3, 5.

Then $h^2+hk+k^2=\frac{3}{2}(h^2+k^2)-\frac{H^2}{2}$.

$=3R^2-\frac{3}{2}(r^2+r'^2)-\frac{H^2}{2}$.

$\therefore V=\frac{1}{3}\pi H\left[\frac{3}{2}(r^2+r'^2)+\frac{H^2}{2}\right]$. Ax. 8.

$\therefore V=\frac{1}{2}(\pi r^2+\pi r'^2)H+\frac{1}{6}\pi H^3$.

Q. E. D.

838. Formula for volume of a spherical segment of one base. In a spherical segment of one base $r'=o$, and $r^2=(2R-H)H$ (Art. 343).

Substituting for r and r' these values in the result of Art. 837,

$$V=\pi H^2\left(R-\frac{H}{3}\right).$$

839. Advantage of measurement formulas. The student should observe carefully that, by the results obtained in Book IX, the measurement of the areas of certain curved surfaces is reduced to the far simpler work of the measurement of the lengths of one or more straight lines; in like manner the measurement of certain volumes bounded by a curved surface is reduced to the simpler work of linear measurements. A similar remark applies to the results of Book VIII.

Ex. 1. Find the volume of a sphere whose radius is 7 in.

Ex. 2. Find the volume of a sphere whose diameter is 7 in.

Ex. 3. In a sphere whose radius is 8 in., find the volume of a spherical segment of one base whose altitude is 3.

EXERCISES. GROUP 72

THEOREMS CONCERNING THE SPHERE

Ex. 1. Of circles of a sphere whose planes pass through a given point within a sphere, the smallest is that circle whose plane is perpendicular to the diameter through the given point.

Ex. 2. If a point on the surface of a given sphere is equidistant from three points on a given small circle of the sphere, it is the pole of the small circle.

Ex. 3. If two sides of a spherical triangle are quadrants, the third side measures the angle opposite that side in the triangle.

Ex. 4. If a spherical triangle has one right angle, the sum of its other two angles is greater than one right angle.

Ex. 5. If a spherical triangle is isosceles, its polar triangle is isosceles.

Ex. 6. The polar triangle of a birectangular triangle is birectangular.

Ex. 7. The polar triangle of a trirectangular triangle is identical with the original triangle.

Ex. 8. Prove that the sum of the angles of a spherical quadrilateral is greater than 4 right angles, and less than 8 right angles. What, also, are the limits of the sum of the angles of a spherical hexagon? Of the sum of the angles of a spherical n-gon?

Ex. 9. On the same sphere, or on equal spheres, two birectangular triangles are equal if their oblique angles are equal.

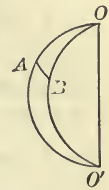
Ex. 10. Two zones on the same sphere, or on equal spheres, are to each other as their altitudes.

Ex. 11. If one of the legs of a right spherical triangle is greater than a quadrant, another side is also greater than a quadrant.

[SUG. Of the leg which is greater than a quadrant, take the end remote from the right angle as a pole, and describe an arc.]

Ex. 12. If ABC and $A'B'C'$ are polar triangles, the radius OA is perpendicular to the plane $OB'C'$.

Ex. 13. On the same sphere, or on equal spheres, spherical triangles whose polar triangles have equal perimeters are equivalent.



Ex. 14. Given OAO', OBO', and AB arcs of great circles, intersecting so that $\angle OAB = \angle O'BA$; prove that $\triangle OAB = \triangle O'AB$.

[Sug. Show that $\angle OBA = \angle O'AB$.]

Ex. 15. Find the ratio of the volume of a sphere to the volume of a circumscribed cube.

Ex. 16. Find the ratio of the surface of a sphere to the lateral surface of a circumscribed cylinder of revolution; also find the ratio of their volumes.

Ex. 17. If the edge of a regular tetrahedron is denoted by a, find the ratio of the volumes of the inscribed and circumscribed spheres.

Ex. 18. Find the ratio of the two segments into which a hemisphere is divided by a plane parallel to the base of the hemisphere and at the distance $\frac{2}{3}R$ from the base.

EXERCISES. GROUP 73

SPHERICAL LOCI

Ex. 1. Find the locus of a point at a given distance a from the surface of a given sphere.

Ex. 2. Find the locus of a point on the surface of a sphere that is equidistant from two given points on the surface.

Ex. 3. If, through a given point outside a given sphere, tangent planes to the sphere are passed, find the locus of the points of tangency.

Ex. 4. If straight lines be passed through a given fixed point in space, and through another given point other straight lines be passed perpendicular to the first set, find the locus of the feet of the perpendiculars.

EXERCISES. GROUP 74

PROBLEMS CONCERNING THE SPHERE

Ex. 1. At a given point on a sphere, construct a plane tangent to the sphere.

Ex. 2. Through a given point on the surface of a sphere, draw an arc of a great circle perpendicular to a given arc.

Ex. 3. Inscribe a circle in a given spherical triangle.

Ex. 4. Construct a spherical triangle, given its polar triangle.

Given the radius, *r*, construct a spherical surface which shall pass through

Ex. 5. Three given points.

Ex. 6. Two given points and be tangent to a given plane.

Ex. 7. Two given points and be tangent to a given sphere

Ex. 8. One given point and be tangent to two given planes.

Ex. 9. One given point and be tangent to two given spheres.

Given the radius, *r*, construct a spherical surface which shall be tangent to

Ex. 10. Three given planes.

Ex. 11. Two given planes and one given sphere.

Ex. 12. Construct a spherical surface which shall pass through three given points and be tangent to a given plane.

Ex. 13. Through a given straight line pass a plane tangent to a given sphere.

[Sug. Through the center of the sphere pass a plane ⊥ given line, etc.]

When is the solution impossible?

Ex. 14. Through a given point on a sphere, construct an arc of a great circle tangent to a given small circle of the sphere.

[Sug. Draw a straight line from the center of the sphere to the given point, and produce it to intersect the plane of the small circle, etc.]

NUMERICAL EXERCISES IN SOLID GEOMETRY

For methods of facilitating numerical computations, see Arts. 493-6.

EXERCISES. GROUP 75

LINES AND SURFACES OF POLYHEDRONS

Find the lateral area and total area of a right prism whose

Ex. 1. Base is an equilateral triangle of edge 4 in., and whose altitude is 15 in.

Ex. 2. Base is a triangle of sides 17, 12, 25, and whose altitude is 20.

Ex. 3. Base is an isosceles trapezoid, the parallel bases being 10 and 15 and leg 8, and whose altitude is 24.

Ex. 4. Base is a rhombus whose diagonals are 12 and 16, and whose altitude is 12.

Ex. 5. Base is a regular hexagon with side 8 ft., and whose altitude is 20 ft.

Ex. 6. Find the entire surface of a rectangular parallelopiped $8 \times 12 \times 16$ in.; of one $p \times q \times r$ ft.

Ex. 7. Of a cube whose edge is 1 ft. 3 in.

Ex. 8. The lateral area of a regular hexagonal prism is 120 sq. ft. and an edge of the base is 10 ft. Find the altitude.

Ex. 9. How many square feet of tin are necessary to line a box $20 \times 6 \times 4$ in.?

Ex. 10. If the surface of a cube is 1 sq. yd., find an edge in inches.

Ex. 11. Find the diagonal of a cube whose edge is 5 in.

Ex. 12. If the diagonal of a cube is 12 ft., find the surface.

Ex. 13. If the surface of a rectangular parallelopiped is 208 sq. in., and the edges are as 2 : 3 : 4, find the edges.

In a regular square pyramid

Ex. 14. If an edge of the base is 16 and slant height is 17, find the altitude.

Ex. 15. If the altitude is 15 and a lateral edge is 17, find an edge of the base.

Ex. 16. If a lateral edge is 25 and an edge of the base is 14, find the altitude.

In a regular triangular pyramid

Ex. 17. If an edge of the base is 8 and the altitude is 10, find the slant height.

Ex. 18. Find the altitude of a regular tetrahedron whose edge is 6.

Find the lateral surface and the total surface of

Ex. 19. A regular square pyramid an edge of whose base is 16, and whose altitude is 15.

Ex. 20. A regular triangular pyramid an edge of whose base is 10, and whose altitude is 12.

Ex. 21. A regular hexagonal pyramid an edge of whose base is 4, and whose altitude is 21.

Ex. 22. A regular square pyramid whose slant height is 24, and whose lateral edge is 25.

Ex. 23. A regular tetrahedron whose edge is 4.

Ex. 24. A regular tetrahedron whose altitude is 9.

Ex. 25. A regular hexagonal pyramid each edge of whose base is a, and whose altitude is b.

Ex. 26. In the frustum of a regular square pyramid the edges of the bases are 6 and 18, and the altitude is 8. Find the slant height. Hence find the lateral area.

Ex. 27. In the frustum of a regular triangular pyramid the edges of the bases are 4 and 6, and the altitude is 5. Find the slant height. Hence find the lateral area.

Ex. 28. In the frustum of a regular tetrahedron, if the edge of the lower base is b_1, the edge of the upper base is b_2, and the altitude is a, show that $L = \frac{1}{2}\sqrt{\frac{1}{3}(b_1 - b_2)^2 + 4a^2}$.

Ex. 29. In the frustum of a regular square pyramid the edges of the bases are 20 and 60, and a lateral edge is 101. Find the lateral surface.

EXERCISES. GROUP 76

LINES AND SURFACES OF CONES AND CYLINDERS

Ex. 1. How many square feet of lateral surface has a tunnel 100 yds. long and 7 ft. in diameter.

Ex. 2. The lateral area of a cylinder of revolution is 1 sq. yd., and the altitude is 1 ft. Find the radius of the base.

Ex. 3. The entire surface of a cylinder of revolution is 900 sq. ft. and the radius of the base is 10 ft. Find the altitude.

In a cylinder of revolution

Ex. 4. Find R in terms of S and H.

Ex. 5. Find H in terms of R and T.

Ex. 6. Find T in terms of S and H.

Ex. 7. How many sq. yds. of canvas are required to make a conical tent 20 ft. in diameter and 12 ft. high?

Ex. 8. A man has 400 sq. yds. of canvas and wants to make a conical tent 20 yds. in diameter. What will be its altitude?

Ex. 9. The altitude of a cone of revolution is 10 ft. and the lateral area is 11 times the area of the base. Find the radius of the base.

In a cone of revolution

Ex. 10. Find T in terms of S and L.

Ex. 11. Find R in terms of T and L.

Ex. 12. How many square feet of tin are necessary to make a funnel the diameters of whose ends are 2 in. and 8 in., and whose altitude is 7 in.?

Ex. 13. If the slant height of a frustum of a cone of revolution makes an angle of 45° with the base, show that the lateral area of the frustum is $(r_1^2 - r_2^2)\pi\sqrt{2}$.

EXERCISES. GROUP 77

SPHERICAL LINES AND SURFACES

Ex. 1. Find in square feet the area of the surface of a sphere whose radius is 1 ft. 2 in.

Ex. 2. How many square inches of leather will it take to cover a baseball whose diameter is $3\frac{1}{2}$ in.?

Ex. 3. How many sq. ft. of tin are required to cover a dome in the shape of a hemisphere 6 yds. in diameter?

Ex. 4. What is the radius of a sphere whose surface is 616 sq. in.?

Ex. 5. Find the diameter of a globe whose surface is 1 sq. yd.

Ex. 6. If the circumference of a great circle on a sphere is 1 ft., find the area of the surface of the sphere.

Ex. 7. If a hemispherical dome is to contain 100 sq. yds. of surface, what must its diameter be?

Ex. 8. Find the radius of a sphere in which the area of the surface equals the number of linear units in the circumference of a great circle.

Find the area of a lune in which

Ex. 9. The angle of the lune is 36°, and the radius of the sphere is 14 in.

Ex. 10. The angle of the lune is 18° 20′, and the diameter of the sphere is 20 in.

Ex. 11. The angle of the lune is 24°, and the surface of the sphere is 4 sq. ft.

Find the area of a spherical triangle in which

Ex. 12. The angles are 80°, 90°, 120°, and the diameter of the sphere is 14 ft.

Ex. 13. The angles are 74° 24′, 83° 16′, 92° 20′, and the radius of the sphere is 10.

Ex. 14. The angles are 85°, 95°, 135°, and the surface of the sphere is 10 sq. ft.

Ex. 15. If the sides of a spherical triangle are 100°, 110°, 120° and the radius of the sphere is 16, find the area of the polar triangle.

Ex. 16. If the angles of a spherical triangle are 90°, 100°, 120° and its area is 3900, find the radius of the sphere.

Ex. 17. If the area of an equilateral spherical triangle is one-third the surface of the sphere, find an angle of the triangle.

Ex. 18. In a trihedral angle the plane angles of the dihedral angles are 80°, 90°, 100°; find the number of solid degrees in the trihedral angle.

Ex. 19. Find the area of a spherical hexagon each of whose angles is 150°, on a sphere whose radius is 20 in.

Ex. 20. If each dihedral angle of a given pentahedral angle is 120°, how many solid degrees does the pentahedral angle contain?

Ex. 21. In a sphere whose radius is 14 in., find the area of a zone 3 in. high.

Ex. 22. What is the area of the north temperate zone, if the earth is taken to be a sphere with a radius of 4,000 miles, and the distance between the plane of the arctic circle and that of the tropic of Cancer is 1,800 miles?

Ex. 23. If Cairo, Egypt, is in latitude 30°, show that its parallel of latitude bisects the surface of the northern hemisphere.

Ex. 24. How high must a person be above the earth's surface to see one-third of the surface?

Ex. 25. How much of the earth's surface will a man see who is 2,000 miles above the surface, if the diameter is taken as 8,000 miles?

Ex. 26. If the area of a zone equals the area of a great circle, find the altitude of the zone in terms of the radius of the sphere.

Ex. 27. If sounds from the Krakatoa explosion were heard at a distance of 3,000 miles (taken as a chord) on the surface of the earth, over what fraction of the earth's surface were they heard?

Ex. 28. The radii of two spheres are 5 and 12 in. and their centers are 13 in. apart. Find the area of the circle of intersection and also of that part of the surface of each sphere not included by the other sphere.

EXERCISES. GROUP 78

VOLUMES OF POLYHEDRONS

Find the volume of a prism

Ex. 1. Whose base is an equilateral triangle with side 5 in., and whose altitude is 16 in.

Ex. 2. Whose base is a triangle with sides 12, 13, 15, and whose altitude is 20.

Ex. 3. Whose base is an isosceles right triangle with a leg equal to 2 yds., and whose altitude is 25 ft.

Ex. 4. Whose base is a regular hexagon with a side of 8 ft., and whose altitude is 10 yds.

Ex. 5. Whose base is a rhombus one of whose sides is 25, and one of whose diagonals is 14, and whose altitude is 11.

Ex. 6. Whose base contains 84 sq. yds., and whose lateral faces are three rectangles with areas of 100, 170, 210 sq. yds., respectively.

Ex. 7. How many bushels of wheat are held by a bin $30 \times 10 \times 6$ ft., if a bushel is taken as $1\frac{1}{4}$ cu. ft.?

Ex. 8. How many cart-loads of earth are in a cellar $30 \times 20 \times 6$ ft., if a cart-load is a cubic yard?

Ex. 9. If a cubical block of marble costs \$2, what is the cost of a cube whose edge is a diagonal of the first block?

Ex. 10. Find the edge of a cube whose volume equals the sum of the volumes of two cubes whose edges are 3 and 5 ft.

Ex. 11. Find the edge of a cube whose volume equals the area of its surface.

Ex. 12. If the top of a cistern is a rectangle 12×8 ft., how deep must the cistern be to hold 10,000 gallons?

Ex. 13. Find the inner edge of a peck measure which is in the shape of a cube.

Ex. 14. A peck measure is to be a rectangular parallelopiped with square base and altitude equal to twice the edge of the base. Find its dimensions.

Ex. 15. Find the volume of a cube whose diagonal is a.

Find the volume of a pyramid

Ex. 16. Whose base is an equilateral triangle with side 8 in., and whose altitude is 12 in.

Ex. 17. Whose base is a right triangle with hypotenuse 29 and one leg 21, and whose altitude is 20.

Ex. 18. Whose base is a square with side 6, and each of whose lateral edges is 5.

Ex. 19. Whose base is a square with side 10, and each of whose lateral faces makes an angle of 45° with the base.

Ex. 20. If the pyramid of Memphis has an altitude of 146 yds. and a square base of side 232 yds., how many cubic yards of stone does it contain? What is this worth at $1 a cu. yd.?

Ex. 21. A church spire 150 ft. high is hexagonal in shape and each side of the base is 10 ft. The spire has a hollow hexagonal interior, each side of whose base is 6 ft., and whose altitude is 45 ft. How many cubic yards of stone does the spire contain?

Ex. 22. If a pyramid contains 4 cu. yds. and its base is a square with one side 2 ft., find the altitude.

Ex. 23. A heap of candy in the shape of a frustum of a regular square pyramid has the edges of its bases 25 and 9 in. and its altitude 12 in. Find the number of pounds in the heap if a pound is a rectangular parallelopiped 4 x 3 x 2 in. in size.

Ex. 24. Find the volume of a frustum of a regular triangular pyramid, the edges of the bases being 2 and 8, and the slant height 12.

Ex. 25. The edges of the bases of the frustum of a regular square pyramid are 24 and 6, and each lateral edge is 15; find the volume.

Ex. 26. If a stick of timber is in the shape of a frustum of a regular square pyramid with the edges of its ends 9 and 15 in., and with a length of 14 ft., find the number of feet of lumber in the stick.

What is the difference between this volume and that of a stick of the same length having the shape of a prism with a base equal to the area of a midsection of the first stick?

Ex. 27. Find the volume of a prismoid whose bases are rectangles 5×2 ft. and 7×4 ft., and whose altitude is 12 ft.

Ex. 28. How many cart-loads of earth are there in a railroad cut 12 ft. deep, whose base is a rectangle 100×8 ft., and whose top is a rectangle 30×50 ft.?

Ex. 29. Find the volume of a prismatoid whose base is an equilateral triangle with side 12 ft., and whose top is a line 12 ft. long parallel to one side of the base, and whose altitude is 15 ft.

Ex. 30. If the base of a prismatoid is a rectangle with dimensions a and b, the top is a line c parallel to the side b of the base, and the altitude is h, find the volume.

EXERCISES. GROUP 79

VOLUMES OF CONES AND CYLINDERS

Ex. 1. How many barrels of oil are contained in a cylindrical tank 20 ft. long and 6 ft. in diameter, if a barrel contains 4 cu. ft.?

X **Ex. 2.** How many cu. yds. of earth must be removed in making a tunnel 450 ft. long, if a cross-section of the tunnel is a semicircle of 15 ft. radius?

X **Ex. 3.** A cylindrical glass 3 in. in diameter holds half a pint. Find its height in inches.

X **Ex. 4.** If a cubic foot of brass be drawn out into wire $\frac{1}{30}$ inch in diameter, how long will the wire be?

Ex. 5. A gallon measure is a cylinder whose altitude equals the diameter of the base. Find the altitude.

Ex. 6. Show that the volumes of two cylinders, having the altitude of each equal to the radius of the other, are to each other as $R : R'$.

Ex. 7. In a cylinder, find R in terms of V and H; also V in terms of S and R.

X **Ex. 8.** A conical heap of potatoes is 44 ft. in circumference and 6 ft. high. How many bushels does it contain, if a bushel is $1\frac{1}{4}$ cu. ft.?

X **Ex. 9.** What fraction of a pint will a conical wine-glass hold, if its altitude is 3 in. and the diameter of the top is 2 in.?

Ex. 10. Find the ratio of the volumes of the two cones inscribed in, and circumscribed about, a regular tetrahedron.

Ex. 11. If an equilateral cone contains 1 quart, find its dimensions in inches.

Ex. 12. In a cone of revolution find V in terms of R and L; also find V in terms of R and S.

Ex. 13. Find the volume of a frustum of a cone of revolution, whose radii are 14 and 7 ft., and whose altitude is 2 yds.

Ex. 14. What is the cost, at 50 cts. a cu. ft., of a piece of marble in the shape of a frustum of a cone of revolution, whose radii are 6 and 9 ft., and whose slant height is 5 ft.?

Ex. 15. In a frustum of a cone of revolution, the volume is 88 cu. ft., the altitude is 9 ft., and $R=2r$. Find r.

EXERCISES. GROUP 80

SPHERICAL VOLUMES

Ex. 1. Find the volume of a sphere whose radius is 1 ft. 9 in.

Ex. 2. If the earth is a sphere 7,920 miles in diameter, find its volume in cubic miles.

Ex. 3. Find the diameter of a sphere whose volume is 1 cu. ft.

Ex. 4. What is the volume of a sphere whose surface is 616 sq. in.?

Ex. 5. Find the radius of a sphere equivalent to the sum of two spheres, whose radii are 2 and 4 in.

Ex. 6. Find the radius of a sphere whose volume equals the area of its surface.

Ex. 7. Find the volume of a sphere circumscribed about a cube whose edge is 6.

Ex. 8. Find the volume of a spherical shell whose inner and outer diameters are 14 and 21 in.

Ex. 9. Find the volume of a spherical shell whose inner and outer surfaces are 20π and 12π.

Find the volume of

Ex. 10. A spherical wedge whose angle is 24°, the radius of the sphere being 10 in.

Ex. 11. A spherical sector whose base is a zone 2 in. high, the radius of the sphere being 10 in.

Ex. 12. A spherical segment of two bases whose radii are 4 and 7 and altitude 5 in.

Ex. 13. A wash-basin in the shape of a segment of a sphere is 6 in. deep and 24 in. in diameter. How many quarts of water will the basin hold ?

Ex. 14. A plane parallel to the base of a hemisphere and bisecting the altitude divides its volume in what ratio ?

Ex. 15. A spherical segment 4 in. high contains 200 cu. in.; find the radius of the sphere.

Ex. 16. If a heavy sphere whose diameter is 4 in. be placed in a conical wine-glass full of water, whose diameter is 5 in. and altitude 6 in., find how much water will run over.

EXERCISES. GROUP 81

EQUIVALENT SOLIDS

Ex. 1. If a cubical block of putty, each edge of which is 8 inches, be molded into a cylinder of revolution whose radius is 3 inches, find the altitude of the cylinder.

Ex. 2. Find the radius of a sphere equivalent to a cube whose edge is 10 in.

Ex. 3. Find the radius of a sphere equivalent to a cone of revolution, whose radius is 3 in. and altitude 6 in.

Ex. 4. Find the edge of a cube equivalent to a frustum of a cone of revolution, whose radii are 4 and 9 ft. and altitude 2 yds.

Ex. 5. Find the altitude of a rectangular parallelopiped, whose base is 3 x 5 in. and whose volume is equivalent to a sphere of radius 7 in.

Ex. 6. Find the base of a square rectangular parallelopiped, whose altitude is 8 in. and whose volume equals the volume of a cone of revolution with a radius of 6 and an altitude of 12 in.

Ex. 7. Find the radius of a cone of revolution, whose altitude is 15 and whose volume is equal to that of a cylinder of revolution with radius 6 and altitude 20.

Ex. 8. Find the altitude of a cone of revolution, whose radius is 15 and whose volume equals the volume of a cone of revolution with radius 9 and altitude 24.

Ex. 9. On a sphere whose diameter is 14 the altitude of a zone of one base is 2. Find the altitude of a cylinder of revolution, whose base equals the base of the zone and whose lateral surface equals the surface of the zone.

480 - Page

EXERCISES. GROUP 82

SIMILAR SOLIDS

Ex. 1. If on two similar solids L, L' and l, l' are pairs of homologous lines; A, A' and a, a' pairs of homologous areas, V, V' and v, v' pairs of homologous volumes,

$$\text{show that } \begin{cases} L : L' = l : l' = \sqrt{A} : \sqrt{A'} = \sqrt[3]{v} : \sqrt[3]{v'}. \\ A : A' = L^2 : L'^2 = a : a' = V^{\frac{2}{3}} : V'^{\frac{2}{3}}. \\ V : V' = L^3 : L'^3 = A^{\frac{3}{2}} : A'^{\frac{3}{2}} = v : v'. \end{cases}$$

Ex. 2. If the edge of a cube is 10 in., find the edge of a cube having 5 times the surface.

Ex. 3. If the radius of a sphere is 10 in., find the radius of a sphere having 5 times the surface.

Ex. 4. If the altitude of a cone of revolution is 10 in., find the altitude of a similar cone of revolution having 5 times the surface.

Ex. 5. In the last three exercises, find the required dimension if the volume is to be 5 times the volume of the original solid.

Ex. 6. The linear dimensions of one trunk are twice as great as those of another trunk. How much greater is the volume? The surface?

4400 : 21 = 209.5

Ex. 7. How far from the vertex is the cross-section which bisects the volume of a cone of revolution? Which bisects the lateral surface?

Ex. 8. If the altitude of a pyramid is bisected by a plane parallel to the base, how does the area of the cross-section compare with the area of the base? How does the volume cut off compare with the volume of the entire pyramid?

Ex. 9. Planes parallel to the base of a cone divide the altitude into three equal parts; compare the lateral surfaces cut off. Also the volumes.

Ex. 10. A sphere 10 in. in diameter is divided into three equivalent parts by concentric spherical surfaces. Find the diameters of these surfaces.

Ex. 11. If the strength of a muscle is as the area of its cross-section, and Goliath of Gath was three times as large in each linear dimension as Tom Thumb, how much greater was his strength? His weight? How, then, does the activity of the one man compare with that of the other?

Ex. 12. If the rate at which heat radiates from a body is in proportion to the amount of surface, and the planet Jupiter has a diameter 11 times that of the earth, how many times longer will Jupiter be in cooling off?

[Sug. How many times greater is the volume, and therefore the original amount of heat in Jupiter? How many times greater is its surface? What will be the combined effect of these factors?]

GROUP 83

MISCELLANEOUS NUMERICAL EXERCISES IN SOLID GEOMETRY

Find S, T and V of

Ex. 1. A right triangular prism whose altitude is 1 ft., and the sides of whose base are 26, 28, 30 in.

Ex. 2. A cone of revolution the radius of whose base is 1 ft. 2 in., and whose altitude is 35 in.

Ex. 3. A frustum of a square pyramid the areas of whose bases are 1 sq. ft. and 36 sq. in., and whose altitude is 9 in.

Ex. 4. A pyramid whose slant height is 10 in., and whose base is an equilateral triangle whose side is 8 in.

Ex. 5. A cube whose diagonal is 1 yd.

Ex. 6. A frustum of a cone of revolution whose radii are 6 and 11 in. and slant height 13 in.

Ex. 7. A rectangular parallelopiped whose diagonal is $2\sqrt{29}$, and whose dimensions are in the ratio 2 : 3 : 4.

Ex. 8. Find the volume of a sphere inscribed in a cube whose edge is 6; also find the area of a triangle on that sphere whose angles are 80°, 90°, 150°.

Ex. 9. Find the volume of the spherical pyramid whose base is the above triangle.

Ex. 10. Find the angle of a lune on the same sphere, equivalent to that triangle.

Ex. 11. On a cube whose edge is 4, planes through the midpoints of the edges cut off the corners. Find the volume of the solid remaining.

Ex. 12. How is V changed if H of a cone of revolution is doubled and R remains unchanged? If R is doubled and H remains unchanged? If both H and R are doubled?

Ex. 13. In an equilateral cone, find S and V in terms of R.

Ex. 14. A piece of lead 20 x 8 x 2 in. will make how many spherical bullets, each $\frac{3}{4}$ in. in diameter?

Ex. 15. How many bricks are necessary to make a chimney in the shape of a frustum of a cone, whose altitude is 90 ft., whose outer diameters are 3 and 8 ft., and whose inner diameters are 2 and 4 ft., counting 12 bricks to the cubic ft.?

Ex. 16. If the area of a zone is 300 and its altitude 6, find the radius of the sphere.

Ex. 17. If the section of a cylinder of revolution through its axis is a square, find S, T, V in terms of R.

Ex. 18. If every edge of a square pyramid is b, find b in terms of T.

Ex. 19. A regular square pyramid has a for its altitude and also for each side of the base. Find the area of a section made by a plane parallel to the base and bisecting the altitude. Find also the volumes of the two parts into which the pyramid is divided.

Ex. 20. If the earth is a sphere of 8,000 miles diameter and its atmosphere extends 50 miles from the earth, find the volume of the atmosphere.

Ex. 21. On a sphere, find the ratio of the area of an equilateral spherical triangle, each of whose angles is 95°, to the area of a lune whose angle is 80°.

Ex. 22. A square right prism has an altitude $6a$ and an edge of the base $2a$. Find the volume of the largest cylinder, sphere, pyramid and cone which can be cut from it.

Ex. 23. Obtain a formula for the area of that part of a sphere illuminated by a point of light at a distance a from the sphere whose radius is R.

Ex. 24. On a sphere whose radius is 6 in., find an angle of an equilateral triangle whose area is 12 sq. in.

Ex. 25. Find the volume of a prismatoid, whose altitude is 24 and whose bases are equilateral triangles, each side 10, so placed that the mid-section of the prismatoid is a regular hexagon.

Ex. 26. On a sphere whose radius is 16, the bases of a zone are equal and are together equal to the area of the zone. Find the altitude of the zone.

Ex. 27. Find the volume of a square pyramid, the edge of whose base is 10 and each of whose lateral edges is inclined 60° to the base.

Ex. 28. An irregular piece of ore, if placed in a cylinder partly filled with water, causes the water to rise 6 in. If the radius of the cylinder is 8 in., what is the volume of the ore?

Ex. 29. Find the volume of a truncated right triangular prism, if the edges of the base are 8, 9, 11, and the lateral edges are 12, 13, 14.

Ex. 30. In a sphere whose radius is 5, a section is taken at the distance 3 from the center. On this section as a base a cone is formed whose lateral elements are tangent to the sphere. Find the lateral surface and volume of the cone.

Ex. 31. The volume of a sphere is $1{,}437\frac{1}{3}$ cu. in. Find the surface.

Ex. 32. A square whose side is 6 is revolved about a diagonal as an axis; find the surface and volume generated.

Ex. 33. Find the edge of a cubical cistern that will hold 10 tons of water, if 1 cu. ft. of water weighs 62.28 lbs.

Ex. 34. A water trough has equilateral triangles, each side 3 ft., for ends, and is 18 ft. long. How many buckets of water will it hold, if a bucket is a cylinder 1 ft. in diameter and $1\frac{1}{2}$ ft. high?

Ex. 35. The lateral area of a cylinder of revolution is 440 sq. in., and the volume is 1,540 cu. in. Find the radius and altitude.

Ex. 36. The angles of a spherical quadrilateral are 80°, 100°, 120°, 120°. Find the angle of an equivalent equilateral triangle.

Ex. 37. A cone and a cylinder have equal lateral surfaces, and their axis sections are equilateral. Find the ratio of their volumes.

Ex. 38. A water-pipe $\frac{3}{4}$ in. in diameter rises 13 ft. from the ground. How many quarts of water must be drawn from it before the water from under the ground comes out? If a quart runs out in 5 seconds, how long must the water run?

Ex. 39. A cube immersed in a cylinder partly filled with water causes the water to rise 4 in. If the radius of the cylinder is 6 in., what is an edge of the cube?

Ex. 40. An auger hole whose diameter is 3 in. is bored through the center of a sphere whose diameter is 8 inches. Find the volume remaining.

Ex. 41. Show that the volumes of a cone, hemisphere, and cylinder of the same base and altitude are as 1 : 2 : 3.

Ex. 42. The volumes of two similar cylinders of revolution are as 8 : 125; find the ratio of their radii. If the radius of the smaller is 10 in., what is the radius of the larger?

Ex. 43. An iron shell is 2 in. thick and the diameter of its outer surface is 28 in. Find its volume.

Ex. 44. The legs of an isosceles spherical triangle each make an angle of 75° with the base. The legs produced form a lune whose area is four times the area of the triangle. Find the angle of the lune.

$4x^2 - x^2$

$3x^2$

GROUP 84

EXERCISES INVOLVING THE METRIC SYSTEM

Find S, T, V of

Ex. 1. A right prism the edges of whose base are 6 m., 70 dm., 900 cm., and whose altitude is 90 dm.

Ex. 2. A regular square pyramid an edge of whose base is 30 dm., and whose altitude is 1.7 m.

Ex. 3. A sphere whose radius is 0.02 m.

Ex. 4. A frustum of a cone of revolution whose radii are 10 dm. and 6 dm., and whose slant height is 50 cm.

Ex. 5. A cube whose diagonal is 12 cm.

Ex. 6. A cylinder of revolution whose radius equals 2 dm., and whose altitude equals the diameter of the base.

Ex. 7. Find the area of a spherical triangle on a sphere whose radius is 0.02 m., if its angles are 110°, 120°, 130°.

Ex. 8. Find the number of square meters in the surface of a sphere, a great circle of which is 50 dm. long.

Ex. 9. How many liters will a cylindrical vessel hold that is 10 dm. in diameter and 0.25 m. high? How many liquid quarts?

Ex. 10. A liter measure is a cylinder whose diameter is half the altitude. Find its dimensions in centimeters.

Ex. 11. Find the surface of a sphere whose volume is 1 cu. m.

APPENDIX

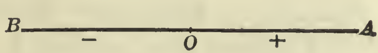
I. MODERN GEOMETRIC CONCEPTS

840. Modern Geometry. In recent times many new geometric ideas have been invented, and some of them developed into important new branches of geometry. Thus, the idea of symmetry (see Art. 484, etc.) is a modern geometric concept. A few other of these modern concepts and methods will be briefly mentioned, but their thorough consideration lies beyond the scope of this book.

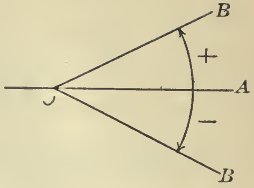
841. Projective Geometry. The idea of projections (see Art. 345) has been developed in comparatively recent times into an important branch of mathematics with many practical applications, as in engineering, architecture, construction of maps, etc.

842. Principle of Continuity. By this principle two or more theorems are made special cases of a single more general theorem. An important aid in obtaining continuity among geometric principles is the application of the concept of negative quantity to geometric magnitudes.

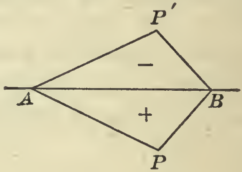
Thus, a negative line is a line opposite in direction to a given line taken as positive.



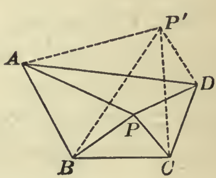
For example, if OA is $+$, OB is $-$.



Similarly, a negative angle is an angle formed by rotating a line in a plane in a direction opposite from a direction of rotation taken as positive. Thus, if the line OA rotating from the position OA forms the positive angle AOB, the same line rotating in the opposite direction forms the negative angle AOB'. Similarly, positive and negative arcs are formed.



In like manner, if P and P' are on opposite sides of the line AB and the area PAB is taken as positive, the area $P'AB$ will be negative.



As an illustration of the law of continuity, we may take the theorem that the sum of the triangles formed by drawing lines from a point to the vertices of a polygon equals the area of the polygon.

Applying this to the quadrilateral $ABCD$, if the point P falls within the quadrilateral, $\triangle PAB + \triangle PBC + \triangle PCD + \triangle PAD = ABCD$ (Ax. 6).

Also, if the point falls without the quadrilateral at P', $\triangle P'AB + \triangle P'BC + \triangle P'CD + \triangle P'AD = ABCD$, since $\triangle P'AD$ is a negative area, and hence is to be subtracted from the sum of the other three triangles.

843. The Principle of Reciprocity, or Duality, is a principle of relation between two theorems by which each theorem is convertible into the other by causing the words for the same two geometric objects in each theorem to exchange places.

Thus, of theorems VI and VII, Book I, either may be converted into the other by replacing the word "sides" by "angles," and "angles" by "sides." Hence these are termed reciprocal theorems.

The following are other instances of reciprocal geometric properties:

1. *Two points determine a straight line.*	1. *Two lines determine a point.*
2. *Three points not in the same straight line determine a plane.*	2. *Three planes not through the same straight line determine a point.*
3. *A straight line and a point determine a plane.*	3. *A straight line and a plane determine a point.*

The reciprocal of a theorem is not necessarily true.

Thus, two parallel straight lines determine a plane, but two parallel planes do not determine a line.

However, by the use of the principle of reciprocity, geometrical properties, not otherwise obvious, are frequently suggested.

844. Principle of Homology. Just as the law of reciprocity indicates relations between one set of geometric concepts (as lines) and another set of geometric concepts (as points), so the law of homology indicates relations between a set of geometric concepts and a set of concepts outside of geometry: as a set of algebraic concepts, for instance.

Thus, if a and b are numbers, by algebra $(a+b)(a-b) = a^2-b^2$.

Also, if a and b are segments of a line, the rectangle $(a+b)\times(a-b)$ is equivalent to the difference between the squares a^2 and b^2.

By means of this principle, truths which would be overlooked or difficult to prove in one department of thought

are made obvious by observing the corresponding truth in another department of thought.

Thus, if a and b are line segments, the theorem $(a+b)^2 + (a-b)^2 = 2(a^2+b^2)$ is not immediately obvious in geometry, but becomes so by observing the like relation between the algebraic numbers a and b.

845. Non-Euclidean Geometry. Hyperspace. By varying the properties of space, as these are ordinarily stated, different kinds of space may be conceived of, each having its own geometric laws and properties. Thus, space, as we ordinarily conceive it, has three dimensions, but it is possible to conceive of space as having four or more dimensions. To mention a single property of four dimensional space, in such a space it would be possible, by simple pressure, to turn a sphere, as an orange, inside out without breaking its surface.

As an aid toward conceiving how this is possible, consider a plane in which one circle lies inside another. No matter how these circles are moved about in the plane, it is impossible to shift the inner circle so as to place it outside the other without breaking the circumference of the outer circle. But, if we are allowed to use the *third* dimension of space, it is a simple matter to lift the inner circle up out of the plane and set it down outside the larger circle.

Similarly if, in space of three dimensions, we have one spherical shell inside a larger shell, it is impossible to place the smaller shell outside the larger without breaking the larger. But if the use of a *fourth* dimension be allowed,—that is, the use of another dimension of freedom of motion,—it is possible to place the inner shell outside the larger without breaking the latter.

846. Curved Spaces. By varying the geometric axioms of space (see Art. 47), different kinds of space may be conceived of. Thus, we may conceive of space such that through a given point *one* line may be drawn parallel to a given line (that is ordinary, or Euclidean space); or such

that through a given point *no line* can be drawn parallel to a given line (spherical space); or such that through a given point *more than one* line can be drawn parallel to a given line (pseudo-spherical space).

These different kinds of space differ in many of their properties. For example, in the first of them the sum of the angles of a triangle equals two right angles; in the second, it is greater; in the third, it is less.

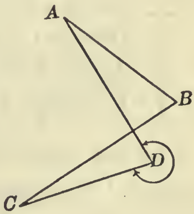
These different kinds of space, however, have many properties in common. Thus, in all of them every point in the perpendicular bisector of a line is equidistant from the extremities of the line.

EXERCISES. GROUP 85

Ex. 1. Show by the use of zero and negative arcs that the principles of Arts. 257, 263, 258, 264, 265, are particular cases of the general theorem that the angle included between two lines which cut or touch a circle is measured by one-half the sum of the intercepted arcs.

Ex. 2. Show that the principles of Arts. 354 and 358 are particular cases of the theorem that, if two lines are drawn from or through a point to meet a circumference, the product of the segments of one line equals the product of the segments of the other line.

Ex. 3. Show by the use of negative angles that theorem XXXVIII, Book I, is true for a quadrilateral of the form *ABCD*. [*BCD* is a negative angle; the angle at the vertex *D* is the reflex angle *ADC*.]



Ex. 4. What is the reciprocal of the statement that two intersecting straight lines determine a plane?

Ex. 5. What is the reciprocal of the statement that three planes perpendicular to each other determine three straight lines perpendicular to each other?

II. HISTORY OF GEOMETRY

847. Origin of Geometry as a Science. The beginnings of geometry as a science are found in Egypt, dating back at least three thousand years before Christ. Herodotus says that geometry, as known in Egypt, grew out of the need of remeasuring pieces of land parts of which had been washed away by the Nile floods, in order to make an equitable readjustment of the taxes on the same.

The substance of the Egyptian geometry is found in an old papyrus roll, now in the British museum. This roll is, in effect, a mathematical treatise written by a scribe named Ahmes at least 1700 B.C., and is, the writer states, a copy of a more ancient work, dating, say, 3000 B. C.

848. Epochs in the Development of Geometry. From Egypt a knowledge of geometry was transferred to Greece, whence it spread to other countries. Hence we have the following principal epochs in the development of geometry:

1. **Egyptian** : 3000 B. C.—1500 B. C.
2. **Greek** : 600 B. C.—100 B. C.
3. **Hindoo** : 500 A. D.—1100 A. D.
4. **Arab** : 800 A. D.—1200 A. D.
5. **European** : 1200 A. D.

In the year 1120 A. D., Athelard, an English monk, visited Cordova, in Spain, in the disguise of a Mohammedan student, and procured a copy of Euclid in the Arabic language. This book he brought back to central Europe, where it was translated into Latin and became the basis of all geometric study in Europe till the year 1533, when,

owing to the capture of Constantinople by the Turks, copies of the works of the Greek mathematicians in the original Greek were scattered through Europe.

HISTORY OF GEOMETRICAL METHODS

849. Rhetorical Methods. By rhetorical methods in the presentation of geometric truths, is meant the use of definitions, axioms, theorems, geometric figures, the representation of geometric magnitudes by the use of letters, the arrangement of material in Books, etc. The **Egyptians** had none of these, their geometric knowledge being recorded only in the shape of the solutions of certain numerical examples, from which the rules used must be inferred.

Thales (Greece 600 B.C.) first made an enunciation of an abstract property of a geometric figure. He had a rude idea of the geometric theorem.

Pythagoras (Italy 525, B. C.) introduced formal definitions into geometry, though some of those used by him were not very accurate. For instance, his definition of a point is "unity having position." Pythagoras also arranged the leading propositions known to him in something like logical order.

Hippocrates (Athens, 420 B. C.) was the first systematically to denote a point by a capital letter, and a segment of a line by two capital letters, as the line AB, as is done at present. He also wrote the first text-book on geometry.

Plato (Athens, 380 B. C.) made definitions, axioms and postulates the beginning and basis of geometry.

To **Euclid** (Alexandria, 280 B. C.) is due the division of geometry into Books, the formal enunciation of theorems, the particular enunciation, the formal construction,

proof, and conclusion, in presenting a proposition. He also introduced the use of the corollary and scholium.

Using these methods of presenting geometric truths, Euclid wrote a text-book of geometry in thirteen books, which was the standard text-book on this subject for nearly two thousand years.

The use of the symbols $\triangle$, ▱, $\parallel$, etc., in geometric proofs originated in the United States in recent years.

850. Logical Methods. The **Egyptians** used no formal methods of proof. They probably obtained their few crude geometric processes as the result of experiment.

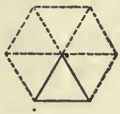
The **Hindoos** also used no formal proof. One of their writers on geometry merely states a theorem, draws a figure, and says "Behold!"

The use of logical methods of geometric proof is due to the **Greeks.** The early Greek geometricians used *experimental methods* at times, in order to obtain geometric truths. For instance, they determined that the angles at the base of an isosceles triangle are equal, by folding half of the triangle over on the altitude as an axis and observing that the angles mentioned coincided as a fact, but without showing that they *must* coincide.

Pythagoras (525 B. C.) was the first to establish geometric truths by systematic *deduction*, but his methods were sometimes faulty. For instance, he believed that the converse of a proposition is necessarily true.

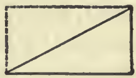
Hippocrates (420 B. C.) used correct and *rigorous deduction* in geometric proofs. He also introduced specific varieties of such deduction, such as the method of *reducing* one proposition to another (Art. 296), and the *reductio ad absurdum*.

The methods of deduction used by the Greeks, however, were defective in their lack of generality. For instance, it was often thought necessary to have a separate proof of a theorem for each different kind of figure to which the theorem applied.



Thus, the theorem that the sum of the angles of a triangle equals two right angles was proved,

(1) for the equilateral triangle by use of the regular hexagon;



(2) for the right triangle by the use of a rectangle;

(3) for a scalene triangle by dividing the scalene triangle into two right triangles.

The Greeks appeared to fear that a general proof might be vitiated if it were applied to a figure in any way special or peculiar. In other words, they had no conception of the principle of continuity (Art. 842).

Plato (380 B. C.) introduced the method of proof by *analysis*, that is, by taking a proposition as true and working from it back to known truths (see Art. 196).

To **Eudoxus** (360 B. C.) is virtually due proof by the method of *limits*; though his method, known as the method of exhaustions, is crude and cumbersome.

Apollonius (Alexandria, 225 B. C.) used *projections*, *transversals*, etc., which, in modern times, have developed into the subject of projective geometry.

851. Mechanical Methods. The **Greeks**, in demonstrating a geometrical theorem, usually drew the figure employed in a bed of sand. This method had certain advantages, but was not adapted to the use of a large audience.

At the time when geometry was being developed in Greece, the interest in the subject was very general. There was scarcely a town but had its lectures on the subject. The news of the discovery of a

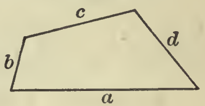
new theorem spread from town to town, and the theorem was redemonstrated in the sand of each market place.

The Greek treatises, however, were written on vellum or papyrus by the use of the reed, or calamus, and ink.

In **Roman times**, and in the **middle ages**, geometrical figures were drawn in wax smeared on wooden boards, called tablets. They were drawn by the use of the stylus, a metal stick, pointed at one end for making marks, and broad at the other for erasing marks. These wax tablets were still in use in Shakespeare's time (see Hamlet Act I, Sc. 5, l. 107). The blackboard and crayon are **modern** inventions, their use having developed within the last one hundred years.

The **Greeks** invented many kinds of drawing instruments for tracing various curves. It was due to the influence of Plato (380 B. C.) that, in constructing geometric figures, the use of only the ruler and compasses is permitted.

HISTORY OF GEOMETRIC TRUTHS. PLANE GEOMETRY



852. Rectilinear Figures. The **Egyptians** measured the area of any four-sided field by multiplying half the sum of one pair of opposite sides by half the sum of the other pair; which was equivalent to using the formula, area $=\frac{a+c}{2}\times\frac{b+d}{2}$.

This, of course, gives a correct result for the rectangle and square, but gives too great a result for other quadrilaterals, as the trapezoid, etc. Hence Joseph, of the Book of Genesis, in buying the fields of the Egyptians for Pharoah in time of famine by the use of this formula, in many cases paid for a larger field than he obtained.

The Egyptians had a special fondness for geometrical constructions, probably growing out of their work as temple

builders. A class of workers existed among them called "rope-stretchers," whose business was the marking out of the foundations of buildings. These men knew how to bisect an angle and also to construct a right angle. The latter was probably done by a method essentially the same as forming a right triangle whose sides are three, four and five units of length. Ahmes, in his treatise, has various constructions of the isosceles trapezoid from different data.

Thales (600 B. C.) enunciated the following theorems:

If two straight lines intersect, the opposite or vertical angles are equal;

The angles at the base of an isosceles triangle are equal;

Two triangles are equal if two sides and the included angle of one are equal to two sides and the included angle of the other;

The sum of the angles of a triangle equals two right angles;

Two mutually equiangular triangles are similar.

Thales used the last of these theorems to measure the height of the great pyramid by measuring the length of the shadow cast by the pyramid and also measuring the length of the shadow of a post of known height at the same time and making a proportion between these quantities.

Pythagoras (525 B. C.) and his followers discovered correct formulas for the areas of the principal rectilinear figures, and also discovered the theorems that the areas of similar polygons are as the squares of their homologous sides, and that the square on the hypotenuse of a right triangle equals the sum of the squares on the other two sides. The latter is called the Pythagorean theorem. They also discovered how to construct a square equivalent to a given parallelogram, and to divide a given line in mean and extreme ratio.

To **Eudoxus** (380 B. C.) we owe the general theory of proportion in geometry, and the treatment of incommensurable quantities by the method of Exhaustions. By the use of these he obtained such theorems as that the areas of two circles are to each other as the squares of their radii, or of their diameters.

In the writings of **Hero** (Alexandria, 125 B. C.) we first find the formula for the area of a triangle in terms of its sides, $K=\sqrt{s(s-a)(s-b)(s-c)}$. Hero also was the first to place land-surveying on a scientific basis.

It is a curious fact that Hero at the same time gives an incorrect formula for the area of a triangle, viz., $K=\frac{1}{2}a(b+c)$, this formula being apparently derived from Egyptian sources.

Xenodorus (150 B. C.) investigated isoperemetrical figures.

The **Romans**, though they excelled in engineering, apparently did not appreciate the value of the Greek geometry. Even after they became acquainted with it, they continued to use antiquated and inaccurate formulas for areas, some of them of obscure origin. Thus, they used the Egyptian formula for the area of a quadrilateral, $K=\frac{a+b}{2}\times\frac{c+d}{2}$. They determined the area of an equilateral triangle whose side is a, by different formulas, all incorrect, as $K=\frac{13a^2}{30}$, $K=\frac{1}{2}(a^2+a)$, and $K=\frac{1}{2}a^2$.

853. The **Circle.** **Thales** enunciated the theorem that every diameter bisects a circle, and proved the theorem that an angle inscribed in a semicircle is a right angle.

To **Hippocrates** (420 B. C.) is due the discovery of nearly all the other principal properties of the circle given in this book.

The **Egyptians** regarded the area of the circle as equivalent to $\frac{64}{81}$ of the diameter squared, which would make $\pi=3.1604$.

The **Jews** and **Babylonians** treated π as equal to 3.

Archimedes, by the use of inscribed and circumscribed regular polygons, showed that the true value of π lies between $3\frac{1}{7}$ and $3\frac{10}{71}$; that is, between 3.14285 and 3.1408.

The **Hindoo** writers assign various values to π, as 3, $3\frac{1}{8}$, $\sqrt{10}$, and **Aryabhatta** (530 A. D.) gives the correct approximation, 3.1416. The Hindoos used the formula $\sqrt{2-\sqrt{4-\overline{AB}^2}}$ (See Art. 468) in computing the numerical value of π.

Within recent times, the value of π has been computed to 707 decimal places.

The use of the symbol π for the ratio of the circumference of a circle to the diameter was established in mathematics by **Euler** (Germany, 1750).

HISTORY OF GEOMETRIC TRUTHS. SOLID GEOMETRY

854. Polyhedrons. The **Egyptians** computed the volumes of solid figures from the linear dimensions of such figures. Thus, Ahmes computes the contents of an Egyptian barn by methods which are equivalent to the use of the formula $V=a\times b\times\frac{3c}{2}$. As the shape of these barns is not known, it is not possible to say whether this formula is correct or not.

Pythagoras discovered, or knew, all the regular polyhedrons except the dodecahedron. These polyhedrons were supposed to have various magical or mystical properties. Hence the study of them was made very prominent.

Hippasus (470 B. C.) discovered the dodecahedron, but he was drowned by the other Pythagoreans for boasting of the discovery.

Eudoxus (380 B. C.) showed that the volume of a pyramid is equivalent to one-third the product of its base by its altitude.

E. F. August (Germany, 1849) introduced the prismatoid formula into geometry and showed its importance.

855. The Three Round Bodies. **Eudoxus** showed that the volume of a cone is equivalent to one-third the area of its base by its altitude.

Archimedes discovered the formulas for the surface and volume of the sphere.

Menelaus (100 A. D.) treated of the properties of spherical triangles.

Gerard (Holland, 1620) invented polar triangles and found the formulas for the area of a spherical triangle and of a spherical polygon.

856. Non-Euclidean Geometry. The idea that a space might exist having different properties from those which we regard as belonging to the space in which we live, has occurred to different thinkers at different times, but **Lobatchewsky** (Russia, 1793-1856) was the first to make systematic use of this principle. He found that if, instead of taking Geom. Ax. 2 as true, we suppose that through a given point in a plane several straight lines may be drawn parallel to a given line, the result is not a series of absurdities or a general reductio ad absurdum; but, on the contrary, a consistent series of theorems is obtained giving the properties of a space.

III. REVIEW EXERCISES

EXERCISES. GROUP 86

REVIEW EXERCISES IN PLANE GEOMETRY

Ex. 1. If the bisectors of two adjacent angles are perpendicular to each other, the angles are supplementary.

Ex. 2. If a diagonal of a quadrilateral bisects two of its angles, the diagonal bisects the quadrilateral.

Ex. 3. Through a given point draw a secant at a given distance from the center of a given circle.

Ex. 4. The bisector of one angle of a triangle and of an exterior angle at another vertex form an angle which is equal to one-half the third angle of the triangle.

Ex. 5. The side of a square is 18 in. Find the circumference of the inscribed and circumscribed circles.

Ex. 6. The quadrilateral $ADBC$ is inscribed in a circle. The diagonals AB and DC intersect in the point F. Arc $AD = 112°$, arc $AC = 108°$, $\angle AFC = 74°$. Find all the other angles of the figure.

Ex. 7. Find the locus of the center of a circle which touches two given equal circles.

Ex. 8. Find the area of a triangle whose sides are 1 m., 17 dm., 210 cm.

Ex. 9. The line joining the midpoints of two radii is perpendicular to the line bisecting their angle.

Ex. 10. If a quadrilateral be inscribed in a circle and its diagonals drawn, how many pairs of similar triangles are formed?

Ex. 11. Prove that the sum of the exterior angles of a polygon (Art. 172) equals four right angles, by the use of a figure formed by drawing lines from a point within a polygon to the vertices of the polygon.

Ex. 12. In a circle whose radius is 12 cm., find the length of the tangent drawn from a point at a distance 240 mm. from the center.

Ex. 13. If two sides of a regular pentagon be produced, find the angle of their intersection.

Ex. 14. In the parallelogram $ABCD$, points are taken on the diagonals such that $AP=BQ=CR=DS$. Show that $PQRS$ is a parallelogram.

Ex. 15. A chord 6 in. long is at the distance 4 in. from the center of a circle. Find the distance from the center of a chord 8 in. long.

Ex. 16. If B is a point in the circumference of a circle whose center is O, PA a tangent at any point P, meeting OB produced at A, and PD perpendicular to OB, then PB bisects the angle APD.

Ex. 17. Construct a parallelogram, given a side, an angle, and a diagonal.

Ex. 18. Find in inches the sides of an isosceles right triangle whose area is 1 sq. yd.

Ex. 19. Given the line a, construct $\frac{a(\sqrt{2}-1)}{3}$.

Ex. 20. If two lines intersect so that the product of the segments of one line equals the product of the segments of the other, a circumference may be passed through the extremities of the two lines.

Ex. 21. Find the locus of the vertices of all triangles on a given base and having a given area.

Ex. 22. On the figure p. 206, prove that $\overline{BC}^2+\overline{AF}^2=\overline{AB}^2+\overline{FC}^2$.

Ex. 23. The area of a rectangle is 108 and the base is three times the altitude. Find the dimensions.

Ex. 24. If, on the sides AC and BC of the triangle ABC, the squares, AD and BF, are constructed, AF and DB are equal.

Ex. 25. If the angle included between a tangent and a secant is half a right angle, and the tangent equals the radius, the secant passes through the center of the circle.

Ex. 26. The sum of the areas of two circles is 20 sq. yds., and the difference of their areas is 15 sq. yds. Find their radii.

Ex. 27. Construct an isosceles trapezoid, given the bases and a leg.

Ex. 28. Show that, if the alternate sides of a regular pentagon be produced to meet, the points of intersection formed are the vertices of another regular pentagon.

Ex. 29. If a post 2 ft. 6 in. high casts a shadow 1 ft. 9 in. long, how tall is a tree which, at the same time, casts a shadow 66 ft. long?

Ex. 30. If two intersecting chords make equal angles with the diameter through their point of intersection, the chords are equal.

Ex. 31. From a given point draw a secant to a circle so that the external segment is half the secant.

Ex. 32. Find the locus of the center of a circle which touches a given circle at a given point.

Ex. 33. If one diagonal of a quadrilateral bisects the other diagonal, the first diagonal divides the quadrilateral into two equivalent triangles.

Ex. 34. In a given square inscribe a square having a given side.

Ex. 35. A field in the shape of an equilateral triangle contains one acre. How many feet does one side contain?

Ex. 36. If perpendiculars are drawn to a given line from the vertices of a parallelogram, the sum of the perpendiculars from two opposite vertices equals the sum of the other two perpendiculars.

Ex. 37. Any two altitudes of a triangle are reciprocally proportional to the bases on which they stand.

Ex. 38. Construct a triangle equivalent to a given triangle and having two given sides.

Ex. 39. The apothem of a regular hexagon is 20. Find the area of the inscribed and circumscribed circles.

Ex. 40. M is the midpoint of the hypotenuse AB of a right triangle ABC. Prove $8\,\overline{MC}^2=\overline{AB}^2+\overline{BC}^2+\overline{AC}^2$.

Ex. 41. Transform a given triangle into an equivalent right triangle containing a given acute angle.

Ex. 42. The area of a square inscribed in a semicircle is to the area of the square inscribed in the circle as 2 : 5.

Ex. 43. If, on a diameter of the circle O, $OA = OB$ and AC is parallel to BD, the chord CD is perpendicular to AC.

Ex. 44. Find the radius of a circle whose area is equal to one-third the area of the circle whose radius is 7 in.

Ex. 45. State and prove the converse of Prop. XXI, Book III.

Ex. 46. If, in a given trapezoid, one base is three times the other base, the segments of each diagonal are as 1 : 3.

Ex. 47. If two sides of a triangle are 6 and 12 and the angle included by them is 60°, find the length of the other side. Also find this when the included angle is 45°; also, when 120°.

Ex. 48. How many sides has a polygon in which the sum of the interior angles exceeds the sum of the exterior angles by 540°?

Ex. 49. If the four sides of a quadrilateral are the diameter of a circle, the two tangents at its extremities, and a tangent at any other point, the area of the quadrilateral equals one-half the product of the diameter by the side opposite it in the quadrilateral.

Ex. 50. An equilateral triangle and a regular hexagon have the same perimeter; find the ratio of their areas.

Ex. 51. To a circle whose radius is 30 cm. a tangent is drawn from a point 21 dm. from the center. Find the length of the tangent.

Ex. 52. If two opposite sides of a quadrilateral are equal, and the angles which they make with a third side are equal, the quadrilateral is a trapezoid.

Ex. 53. If two circles are tangent externally and two parallel diameters are drawn, one in each circle, a pair of opposite extremities of the two diameters and the point of contact are collinear.

Ex. 54. If, in the triangle ABC, the line AD is perpendicular to BD, the bisector of the angle B, a line through D parallel to BC bisects AC.

Ex. 55. Bisect a given triangle by a line parallel to a given line.

Ex. 56. If two parallelograms have an angle of one equal to the supplement of an angle of the other, their areas are to each other as the products of the sides including the angles.

Ex. 57. The sum of the medians of a triangle is less than the perimeter, and greater than half the perimeter.

Ex. 58. If $PARB$ is a secant to a circle through the center O, PT a tangent, and TR perpendicular to PB, then $PA : PR = PO : PB$.

Ex. 59. Two concentric circles have radii of 17 and 15. Find the length of the chord of the larger which is tangent to the smaller.

Ex. 60. On the figure, p. 244,

(*a*) Find two pairs of similar triangles;

(*b*) Find two dotted lines which are perpendicular to each other;

(*c*) Discover a theorem concerning points, not connected by lines on the figure, which are collinear;

(*d*) Discover a theorem concerning squares on given lines.

Ex. 61. One of the legs, AC, of an isosceles triangle is produced through the vertex, C, to the point F, and F is joined with D, the mid-point of the base AB. DF intersects BC in E. Prove that CF is greater than CE.

Ex. 62. The line of centers of two circles intersects their common external tangent at P. $PABCD$ is a secant intersecting one of the two circles at A and B and the other at C and D. Prove $PA \times PD = PB \times PC$.

Ex. 63. Trisect a given parallelogram by lines drawn through a given vertex.

Ex. 64. Find the area of a triangle the sides of which are the chord of an arc of $120°$ in a circle whose radius is 1; the chord of an arc of $90°$ in a circle whose radius is 2; and the chord of an arc of $60°$ in a circle whose radius is 3.

Ex. 65. Construct a triangle, given the median to one side and a median and altitude on the other side.

Ex. 66. Two circles intersect at P and Q. The chord CQ is tangent to the circle QPB at Q. APB is any chord through P. Prove that AC is parallel to BQ.

Ex. 67. In the triangle ABC, from D, the midpoint of BC, DE and DF are drawn, bisecting the angles ADB and ADC, and meeting AB at E and AC at F. Prove $EF \parallel BC$.

Ex. 68. Produce the side BC of the triangle ABC to a point P, so that $PB \times PC = \overline{PA}^2$.

Ex. 69. In a given circle inscribe a rectangle similar to a given rectangle.

Ex. 70. In a given semicircle inscribe a rectangle similar to a given rectangle.

Ex. 71. The area of an isosceles trapezoid is 140 sq. ft., one base is 26 ft., and the legs make an angle of 45° with the other base. Find the other base.

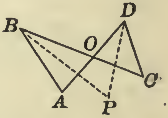
Ex. 72. Cut off one-third the area of a given triangle by a line perpendicular to one side.

Ex. 73. Find the sides of a triangle whose area is 1 sq. ft., if the sides are in the ratio 2 : 3 : 4.

Ex. 74. Divide a given line into two parts such that the sum of the squares of the two parts shall be a minimum.

Ex. 75. If, from any point in the base of a triangle, lines are drawn parallel to the sides, find the locus of the center of the parallelogram so formed.

Ex. 76. Three sides of a quadrilateral are 845, 613, 810, and the fourth side is perpendicular to the sides 845 and 810. Find the area.



Ex. 77. If BP bisects the angle ABC, and DP bisects the angle CDA, prove that angle $P = \frac{1}{2}$ sum of angles A and C.

Ex. 78. Two circles intersect at P and Q. Through a point A in one circumference lines APC and AQD are drawn, meeting the other in C and D. Prove the tangent at A parallel to CD.

Ex. 79. In a given triangle, draw a line parallel to the base and terminated by the sides so that it shall be a mean proportional between the segments of one side.

Ex. 80. Find the angle inscribed in a semicircle the sum of whose sides is a maximum.

Ex. 81. The bases of a trapezoid are 160 and 120, and the altitude 140. Find the dimensions of two equivalent trapezoids into which the given trapezoid is divided by a line parallel to the base.

Ex. 82. If the diameter of a given circle be divided into any two segments, and a semicircumference be described on the two segments on opposite sides of the diameter, the area of the circle will be divided by the semicircumferences thus drawn into two parts having the same ratio as the segments of the diameter.

Ex. 83. On a given straight line, AB, two segments of circles are drawn, APB and AQB. The angles QAP and QBP are bisected by lines meeting in R. Prove that the angle R is a constant, wherever P and Q may be on their arcs.

Ex. 84. On the side AB of the triangle ABC, as diameter, a circle is described. EF is a diameter parallel to BC. Show that EB bisects the angle ABC.

Ex. 85. Construct a trapezoid, given the bases, one diagonal, and an angle included by the diagonals.

Ex. 86. If, through any point in the common chord of two intersecting circles, two chords be drawn, one in each circle, through the four extremities of the two chords a circumference may be passed.

Ex. 87. From a given point as center describe a circle cutting a given straight line in two points, so that the product of the distances of the points from a given point in the line may equal the square of a given line segment.

Ex. 88. AB is any chord in a given circle, P any point on the circumference, PM is perpendicular to AB and is produced to meet the circle at Q; AN is drawn perpendicular to the tangent at P. Prove the triangles NAM and PAQ similar.

Ex. 89. If two circles $ABCD$ and $EBCF$ intersect in B and C and have common exterior tangents AE and DF cut by BC produced at G and H, then $\overline{GH}^2=\overline{BC}^2+\overline{AE}^2$.

EXERCISES. GROUP 87

REVIEW EXERCISES IN SOLID GEOMETRY

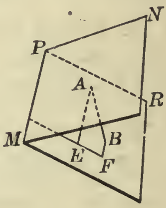
Ex. 1. A segment of a straight line oblique to a plane is greater than its projection on the plane.

Ex. 2. Two tetrahedrons are similar if a dihedral angle of one equals a dihedral angle of the other, and the faces forming these dihedral angles are similar each to each.

Ex. 3. A plane and a straight line, both of which are parallel to the same line, are parallel to each other.

Ex. 4. If the diagonal of one face of a cube is 10 inches, find the volume of the cube.

Ex. 5. Construct a spherical triangle on a given sphere, given the poles of the sides of the triangle.



Ex. 6. Given $AB \perp MN$,
AE and $BF \perp MR$;
prove $EF \perp PM$.

Ex. 7. The diagonals of a rectangular parallelopiped are equal.

Ex. 8. What portion of the surface of a sphere is a triangle each of whose angles is 140°?

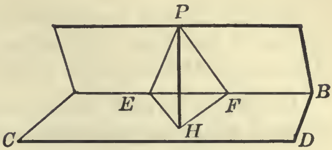
Ex. 9. Through a given point pass a plane parallel to two given straight lines.

Ex. 10. Show that the lateral area of a cylinder of revolution is equivalent to a circle whose radius is a mean proportional between the altitude of the cylinder and the diameter of its base.

Ex. 11. The volumes of polyhedrons circumscribed about equal spheres are to each other as the surfaces of the polyhedrons.

Ex. 12. Find S and T of a regular square pyramid an edge of whose base is 14 dm., and whose lateral edge is 250 cm.

Ex. 13. If two lines are parallel and a plane be passed through each line, the intersection of these planes is parallel to the given lines.



Ex. 14. Given $PH \perp$ plane AD,
$\angle PEH = \angle PFH$;
prove $\angle PEF = \angle PFE$.

Ex. 15. If a plane be passed through the midpoints of three edges of a parallelopiped which meet at a vertex, the pyramid thus formed is what part of the parallelopiped?

Ex. 16. Find a point in a plane such that the sum of its distances from two given points on the same side of the plane is a minimum.

Ex. 17. Given the points A, B, C, D in a plane and P a point outside the plane, AB perpendicular to the plane PBD, and AC perpendicular to the plane PCD; prove that PD is perpendicular to the plane $ABCD$.

Ex. 18. In a sphere whoso radius is 5, find the area of a zone the radii of whose upper and lower bases are 3 and 4.

Ex. 19. Two cylinders of revolution have equal lateral areas. Show that their volumes are as $R : R'$.

Ex. 20. The midpoints of two opposite sides of a quadrilateral in space, and the midpoints of its diagonals, are the vertices of a parallelogram.

Ex. 21. How many feet of two-inch plank are necessary to construct a box twice as wide as deep and twice as long as wide (on the inside), and to contain 216 cu. ft.?

Ex. 22. If two spheres with radii R and r are concentric, find the area of the section of the larger sphere made by a plane tangent to the smaller sphere.

Ex. 23. In the frustum of a regular square pyramid, the edges of the bases are denoted by b_1 and b_2 and the altitude by H; prove that $L = \frac{1}{2}\sqrt{(b_1 - b_2)^2 + 4H^2}$.

Ex. 24. If the opposite sides of a spherical quadrilateral are equal the opposite angles are equal.

Ex. 25. Obtain the simplest formula for the lateral surface of a truncated triangular right prism, each edge of whose base is a, and whose lateral edges are p, q, and r.

Ex. 26. The area of a zone of one base is a mean proportional between the remaining surface of the sphere and its entire surface. Find the altitude of the zone.

Ex. 27. The lateral edges of two similar frusta are as $1 : a$. How do their areas compare? Their volumes?

Ex. 28. Construct a spherical surface with a given radius, r, which shall be tangent to a given plane, and to a given sphere, and also pass through a given point.

Ex. 29. The volume of a right circular cylinder equals the area of the generating rectangle multiplied by the circumference generated by the point of intersection of its diagonals.

Ex. 30. On a sphere whose radius is $8\frac{1}{2}$ inches, find the area of a zone generated by a pair of compasses whose points are 5 inches apart.

Ex. 31. The perpendicular to a given plane from the point where the altitudes of a regular tetrahedron intersect equals one-fourth the sum of the perpendiculars from the vertices of the tetrahedron to the same plane.

Ex. 32. Two trihedral angles are equal or symmetrical if their corresponding dihedral angles are equal.

Ex. 33. On a sphere whose radius is a, a zone has equal bases and the sum of the bases equals the area of the zone. Find the altitude of the zone.

Ex. 34. A plane which bisects two opposite edges of a tetrahedron bisects the volume of the tetrahedron.

Ex. 35. Find the locus of all points in space which have their distances from two given parallel lines in a given ratio.

Ex. 36. If a, b, c are the sides of a spherical triangle, a', b', c' the sides of its polar triangle, and $a>b>c$, then $a'<b'<c'$.

Ex. 37. A cone of revolution has a lateral area of 4 sq. yd. and an altitude of 2 ft. How much of the altitude must be cut off by a plane parallel to the base, in order to leave a frustum whose lateral area is 2 sq. ft.?

Ex. 38. The total area of an equilateral cone is to the area of the inscribed sphere as 9 : 4.

Ex. 39. Construct a sphere of given radius, r, whose surface shall be tangent to three given spheres.

Ex. 40. The volume of the frustum of an equilateral cone is 300 cu. in. and its altitude is 20 in. Show how to find the radii of the bases.

Ex. 41. On each base of a cylinder of revolution a cone is placed, with its vertex at the center of the opposite base. Find the radius of the circle of intersection of the two conical surfaces.

Ex. 42. The volume of a frustum of a cone of revolution equals the sum of a cylinder and a cone of the same altitude as the frustum, and with radii which are respectively the half sum and the half difference of the radii of the frustum.

Ex. 43. A square whose side is a revolves about a line through one of its vertices and parallel to a diagonal, as axis; find the surface and volume generated.

Ex. 44. If a cone of revolution roll on another fixed cone of revolution so that their vertices coincide, find the kind of surface generated by the axis of the rolling cone.

Ex. 45. An equilateral triangle whose side is a revolves about an altitude as an axis; find the surface and volume generated by the inscribed circle, and also by the circumscribed circle.

Ex. 46. Find the locus of the center of a sphere which is tangent to three given planes.

Ex. 47. If an equilateral triangle whose side is a be rotated about a line through one vertex and parallel to the opposite side, as an axis, find the surface and volume generated.

Ex. 48. What other formulas of solid geometry may be regarded as special cases of the formula for the volume of a prismatoid?

Ex. 49. Through a given point pass a plane which shall bisect the volume of a given tetrahedron.

Ex. 50. In an equilateral cone and a cone whose opposite elements are perpendicular at the vertex, show that the ratio of the vertical solid angles is as $2-\sqrt{3} : 2-\sqrt{2}$.

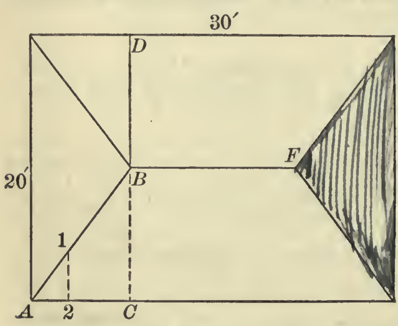
APPLICATIONS OF SOLID GEOMETRY TO MECHANICS AND ENGINEERING

EXERCISES. GROUP 93

(BOOKS VI AND VII)

1. A carpenter tests the flatness of a surface by applying a straight edge to the surface in various directions. How does a plasterer test the flatness of a wall surface? What geometric principle is used by these mechanics?

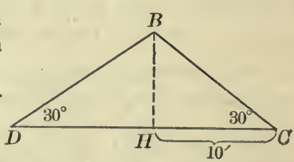
2. Explain why an object with three legs, as a stool or tripod, always rests firmly on the floor while an object with four legs, as a table, does not always rest so. Why do we ever use four-legged pieces of furniture?



3. How can a carpenter get a corner post of a house in a vertical position by use of a carpenter's square? What geometrical principle does he use?

4. The diagram is the plan of a hip roof. The slope of each face of the roof is 30°. Find the length of a hip rafter as AB.

[SUG. Draw a triangle DBC representing a section of the roof at DBC on the plan. Hence it may be shown that $BC = \frac{20}{3}\sqrt{3}$. In like manner by taking a section through BF, it is found that $AC = 10$. Hence in the triangle ABC, AB may be found.]



5. Find the area of the entire roof represented in Ex. 4.

6. Make drawings showing at what angle the two ends of a rafter like BC in Ex. 4 must be cut.

7. Make drawings showing at what angle a jack rafter like 12 in Ex. 4 must be cut.

[SUG. To determine how the end 1 of the jack rafter must be cut use the principle that two intersecting straight lines determine a plane (Art. 501). The cutting plane at 1 must make an angle at the side of the jack rafter equal to angle CBH, and on the top of the jack rafter equal to angle ABC.]

8. What is a gambrel roof? Make up a set of examples concerning a gambrel roof similar to Exs. 4–7.

9. By use of Art. 645, show that a page of this book held at twice the distance of another page from the same lamp receives one fourth the light the first page receives.

10. The supporting power of a wooden beam varies directly as the area of the cross section times the height of the beam and inversely as the length of the beam. Compare the supporting power of a beam 12 ft. long, 3 in. wide, and 6 in. high with that of a beam 18 ft. long, 4 in. wide, and 10 in. high. Also compare the volumes of the two beams.

EXERCISES. GROUP 94

(BOOKS VIII AND IX)

1. A hollow cylinder whose inside diameter is 6 in. is partly filled with water. An irregularly shaped piece of ore when placed in the water causes the top surface of the water to rise 3.4 in. in the cylinder. Find the volume of the ore.

2. What is a tubular boiler? What is the advantage in using a tubular boiler as compared with a plain cylindrical boiler? If a tubular boiler is 18 ft. long and contains 32 tubes each 3 in. in diameter, how much more heating surface has it than a plain cylindrical boiler of the same length and 36 in. in diameter? (Indicate both the long method and the short method of making this computation and use the short method.)

3. If a bridge is to have its linear dimensions 1000 times as great as those of a given model, the bridge will be how many times as heavy as the model?

Why, then, may a bridge be planned so that in the model it will support relatively heavy weights, yet when constructed according to the model, falls to pieces of its own weight?

Show that this principle applies to other constructions, such as buildings, machines, etc., as well as to bridges.

4. Work again Exs. 23–25, 27, p. 473.

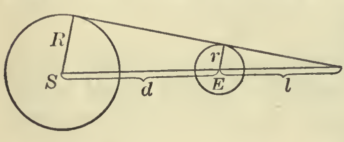
5. Make and work for yourself an example similar to Ex. 24, p. 473.

6. Sound spreads from a center in the form of the surface of an expanding sphere. At the distance of 10 yd. from the source, how will the surface of this sphere compare with its surface as it was at 1 yd.? How, then, does the intensity of sound at 10 yd. from the source compare with its intensity at a distance of 1 yd.?

Does this law apply to all forces which radiate or act from a center as to light, heat, magnetism, and gravitation? Why is it called the law of inverse squares?

7. If a body be placed within a spherical shell, the attractive forces exerted upon the body by different parts of the shell will balance or cancel each other. Hence a body inside the earth, as at the foot of a mine, is attracted effectively only by the sphere of matter whose radius is the distance from the center of the earth to the given body. Hence, prove that the weight of a body below the surface of the earth varies as the distance of the body from the center of the earth.

[Sug. If W denote the weight of the body at the surface, and w its weight when below, R the radius of the earth, and r the distance of the body from the center when below the surface, show that



$$W:w = \frac{R^3}{R^2} : \frac{r^3}{r^2} = R:r$$

8. The light of the sun falling on a smaller sphere, as on the earth or the moon, causes that body to cast a conical shadow. Denoting the radius of the sun by R, the radius of the smaller sphere by r, the distance between the two spheres by d, and the length of the shadow by l, show that $l = \dfrac{d\,r}{R - r}$.

Find l when $d = 92{,}800{,}000$ mi., $R = 433{,}000$ mi., $r = 4000$ mi.

9. If in a lunar eclipse the moon's center should pass through the axis of the conical shadow, and the moon is traveling at the rate of 2100 mi. an hour, how long would the total eclipse of the moon last?

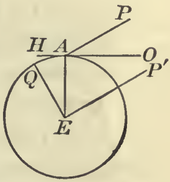
How long if the moon's center passed through the earth's conical shadow at a distance of 1000 mi. from the axis of the cone?

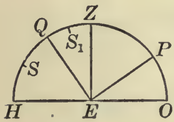
10. If the moon's diameter is 2160 mi., find the length of the moon's shadow as caused by sunlight.

11. If the distance of the moon from the earth's center varies from 221,600 mi. to 252,970, show how this explains why some eclipses of the sun are total and others annular.

Why, also, at a given point on the earth's surface is an eclipse of the sun a so much rarer sight than an eclipse of the moon? Why, also, is its duration so much briefer?

12. Prove that the latitude of a place on the earth's surface equals the elevation of the pole (that is, on the diagram, prove $\angle QEA = \angle PAO$).





13. Given the sun's declination (*i.e.* distance north or south of the celestial equator), show how to determine the latitude of a place by measuring the zenith distance of the sun. Also by measuring the altitude of the sun above the horizon.

14. How was Peary aided by the principles of Exs. 26 and 27 in determining whether he had arrived at the North Pole?

15. If on April 6 (the day of the year on which Peary was at the North Pole) the sun was 6° 7′ north of the celestial equator, how high above the horizon should the sun have been as observed by Peary? At what hour of the day was this?

16. What is the sextant? Explain and prove the principle of the sextant.

17. Explain and prove the principle of the angle-meter.

FORMULAS OF PLANE GEOMETRY

SYMBOLS

a, b, c = sides of triangle ABC.
$s = \frac{1}{2}(a + b + c)$.
h_c = altitude on side c.
m_c = median on side c.
t_c = bisector of angle opposite side c.
l and m = line segments.
P = perimeter.
S_n = side of a regular polygon of n sides.
R = radius of a circle.
D = diameter of a circle.
C = circumference of a circle.
r = radius of an inscribed circle.
$\pi = \frac{22}{7}$ approx. (or 3.1416—).
K = area.
b = base of a triangle.
h = altitude of a triangle.
b_1 and b_2 = bases of a trapezoid.

LENGTHS OF LINES

1. In a right triangle, C being the right angle,

$$c^2 = a^2 + b^2.$$ Art. 346.

2. In a right triangle, l and m being the projections of a and b on c, and h, the altitude on c,

$$h^2 = l \times m.$$
$$a^2 = l \times c, \quad b^2 = m \times c.$$ Art. 342.

3. In an oblique triangle, m being the projection of b on c,
if a is opposite an acute $\angle$, $a^2 = b^2 + c^2 - 2\, c \times m$. Art. 349.
if a is opposite an obtuse $\angle$, $a^2 = b^2 + c^2 + 2\, c \times m$. Art. 350.

4. $h_c = \frac{2}{c}\sqrt{s(s-a)(s-b)(s-c)}$. Art. 393.

5. $m_c = \frac{1}{2}\sqrt{2(a^2 + b^2) - c^2}$. Art. 353.

6. $t_c = \dfrac{2}{a+b}\sqrt{abs(s-c)}$. Art. 363.

7. If l and m are the segments of c made by the bisector of the angle opposite,

$$a : b = l : m.$$ Arts. 332, 336.

8. If l and m are the segments of a line, a, divided in extreme and mean ratio, and $l > m$, $a : l = l : m$. Art. 370.

9. In similar polygons, $\begin{cases} P : P' = a : a'. \\ a : a' = b : b'. \end{cases}$ Art. 341. Art. 321.

10. In circles, $C : C' = R : R'$; also $C : C' = D : D'$. Art. 442.

11. $C = 2\pi R$, or $C = \pi D$. Art. 444.

12. An arc $= \dfrac{\text{central angle}}{180°} \times \pi R$. Art. 445.

13. In inscribed regular polygons,

$$S_{2n} = \sqrt{R\left(2R - \sqrt{4R^2 - S_n^2}\right)}.$$ Art. 467.

AREAS OF PLANE FIGURES

1. In a triangle, $K = \frac{1}{2} bh$. Art. 389.

2. In a triangle, $K = \sqrt{s(s-a)(s-b)(s-c)}$. Art. 393.

3. In an equilateral triangle, $K = \dfrac{a^2\sqrt{3}}{4}$. Ex. 4, p. 257.

4. In a parallelogram, $K = b \times h$. Art. 385.

5. In a trapezoid, $K = \frac{1}{2} h (b_1 + b_2)$. Art. 394.

6. In a regular polygon, $K = \frac{1}{2} r \times P$. Art. 446.

7. In a circle, $K = \pi R^2$ or $K = \frac{1}{4} \pi D^2$. Art. 448.

8. In a sector of a circle, $K = \frac{1}{2} R \times \text{arc}$, Art. 453.

   or $K = \dfrac{\text{central } \angle}{360°} \times \pi R^2$. Art. 453.

9. In a segment of a circle, $K = \text{sector} \pm \triangle$ formed by the chord and radii of the segment.

10. In a circular ring, $K = \pi (R^2 - R'^2)$. Art. 449.

11. In any two similar plane figures,

    $K : K' = a^2 : a'^2$; Art. 399.

    also $a : a' = \sqrt{K} : \sqrt{K'}$. Art. 314.

12. In two circles, $K : K' = R^2 : R'^2 = D^2 : D'^2 = C^2 : C'^2$; Art. 452.

    also $R : R' = D : D' = C : C' = \sqrt{K} : \sqrt{K'}$. Art. 314.

FORMULAS OF SOLID GEOMETRY

SYMBOLS

B, b = areas of the lower and upper bases of a frustum.
E = lateral edge (or element); or = spherical excess.
H = altitude.
l, b, h = length, breadth, height.
L = slant height.
M = area of midsection.
P = perimeter of right section.
P, p = perimeters of lower and upper bases of a frustum.
r, r' = radii of bases.
S = area of lateral surface; or = area of surface of sphere, etc.
T = area of total surface.
V = volume.

FORMULAS FOR AREAS

1. In a prism, $S=E\times P$.	Art. 608.
2. In a regular pyramid, $S=\frac{1}{2}L\times P$.	Art. 641.
3. In a frustum of a regular pyramid, $S=\frac{1}{2}(P+p)L$.	Art. 643.
4. In a cylinder of revolution, $S=2\pi RH$.	Art. 697.
$T=2\pi R(R+H)$.	Art. 697.
5. In a cone of revolution, $S=\pi RL$.	Art. 721.
$T=\pi R(L+R)$.	Art. 721.
6. In a frustum of a cone of revolution, $S=\pi L(R+r)$.	Art. 727.
7. In a sphere, $S=4\pi R^2$, or $S=\pi D^2$.	Art. 810.
8. In a zone, $S=2\pi RH$.	Art. 813.
9. In a lune, $S=\frac{\pi R^2 A}{90}$.	Art. 817.
10. In a spherical triangle, $S=\frac{\pi R^2 E}{180}$.	Art. 822.
11. In a spherical polygon, $S=\frac{\pi R^2 E}{180}$.	Art. 824.

Area of illumination = $\frac{2\pi R^2 a}{R+a}$
distance of light

FORMULAS FOR VOLUMES

1. In a prism, $V = B \times H$. Art. 628.
2. In a parallelopiped, $V = l \times b \times h$. Art. 626.
3. In a pyramid, $V = \frac{1}{3} B \times H$. Art. 651.
4. In a frustum of a pyramid, $V = \frac{1}{3} H (B + b + \sqrt{Bb})$. Art. 656.
5. In a prismatoid, $V = \frac{1}{6} H (B + b + 4M)$. Art. 663.
6. In a cylinder, $V = B \times H$. Art. 698.
7. In a cylinder of revolution, $V = \pi R^2 H$. Art. 699.
8. In a cone, $V = \frac{1}{3} B \times H$. Art. 722.
9. In a circular cone, $V = \frac{1}{3} \pi R^2 H$. Art. 723.
10. In a frustum of a cone, $V = \frac{1}{3} H (B + b + \sqrt{Bb})$. Art. 729.
11. In a frustum of a cone of revolution,
    $V = \frac{1}{3} \pi H (R^2 + r^2 + Rr)$. Art. 730.
12. In a sphere, $V = \frac{4}{3} \pi R^3$, or $V = \frac{1}{6} \pi D^3$. Art. 832.
13. In a spherical sector, $V = \frac{2}{3} \pi R^2 H$. Art. 836.
14. In a spherical segment of two bases,
    $V = \frac{1}{2} (\pi r^2 + \pi r'^2) H + \frac{1}{6} \pi H^3$. Art. 837.
15. In a spherical segment of one base, $V = \pi H^2 (R - \frac{1}{3} H)$.
    Art. 838.

CONSTANTS

1 acre = 43,560 sq. ft.	$\sqrt[3]{2} = 1.2599 +$
1 bushel = 2150.42 cu. in.	$\sqrt[3]{3} = 1.4422 +$
1 gallon = 231 cu. in.	$\frac{1}{\pi} = .3183 +$
$\sqrt{2} = 1.4142 +$	$\sqrt{\pi} = 1.7725 -$
$\sqrt{3} = 1.7321 -$	$\frac{1}{\sqrt{\pi}} = 0.5642 +$

SUMMARY OF METRIC SYSTEM

TABLE FOR LENGTH

10 millimeters (mm.)	=1 centimeter (cm.)
10 cm.	=1 decimeter (dm.)
10 dm.	=1 meter (m.).
10 m.	=1 Dekameter (Dm.)
10 Dm.	=1 Hektometer (Hm.)
10 Hm.	=1 Kilometer (Km.)
10 Km.	=1 Myriameter (Mm.)

Similar tables are used for the unit of weight, the *gram*; for the unit of capacity, the *liter*; for the unit of land measure, the *are*; and for the unit of wood measure, the *stere*.

TABLE FOR SQUARE MEASURE

100 sq. mm. = 1 sq. cm.

100 sq. cm. = 1 sq. dm., etc.

TABLE FOR CUBIC MEASURE

1000 cu. mm. = 1 cu. cm.

1000 cu. cm. = 1 cu. dm., etc.

A *liter* = 1 cu. dm.

A *gram* = weight of 1 cu. cm. of water at 39.2° Fahrenheit.

An *are* = 100 sq. m.

A *stere* = 1 cu. m.

EQUIVALENTS

1 meter	=39.37 inches.
1 liter	=1.057 liquid quarts,
	or .9581 dry quarts.
1 kilogram	=2.2046 lbs. av.
1 hektare	=2.471 acres.
1 sq. m.	= 1550— sq. in.

INDEX OF DEFINITIONS AND FORMULAS